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2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

To Our Colleagues in Mathematics

From the Yosemite in May I bid you come to Baton Rouge in December!

This year's meetings will afford an unusual opportunity for a great gathering of mathematicians from the North and the South. It is a matter to compel our attention when the three large organizations in mathematics, in the research, collegiate and secondary fields, meet together, separating themselves for the once from the other sciences, and join in a week of strong programs in which these fields will overlap. Extensive plans are being formed to insure programs that will appeal to all and that will give our teachers of mathematics the chance to hear and to meet the leaders in American mathematics.

If we believe that there are great advantages in meeting with those who, like ourselves, are vitally eager to maintain and to strengthen the influence of mathematical science and teaching, let us support that belief by planning to make these meetings and the social features connected with them the central portion of our Christmas holidays.

W. D. CAIRNS,
Secretary-Treasurer,
Mathematical Association
of America

The Solution of Numerical Equations

By CLIFFORD BELL
University of California at Los Angeles

The maximum error in any approximation to an irrational root of a numerical equation is of utmost importance, but the common methods of obtaining such a root, unfortunately, do not by themselves give a means of determining this error. Thus, Newton's method, when applied to the ends of an interval within which the root lies, gives approximations in general on the same side of the root. Likewise the approximations obtained by the interpolation method approach the root from one side only. Certain combinations of methods will sometimes furnish a means of determining the accuracy of a given approximation. Thus,* by the use of the two above-mentioned methods simultaneously over a proper interval, the maximum error in either approximation is numerically less than the absolute value of the difference in the approximations.

It would seem desirable to have a single practical method which would give approximations on opposite sides of the root when applied to a proper interval. It is the purpose of this paper to describe such a method.

Assume that the function, $f(x)$, is continuous and possesses derivatives of at least order three. Suppose, further, that the equation, $f(x)=0$, has an irrational root between x_1 and x_2, where $x_2>x_1$, and that none of the derivatives $f'(x)$, $f''(x)$, $f'''(x)$, vanishes in the interval $x_1 \leqq x \leqq x_2$. If x satisfies this condition, the osculatory parabola† drawn to the curve, $y=f(x)$, at the point (x, y), crosses the curve at the point of contact. Let y, Y represent the ordinates of the curve and the parabola, respectively, when x takes on the increment h. For sufficiently small values of h, $y-Y$ has the same sign as $h^3f'''(x)$. Draw the osculatory parabolas at the end points of the curve in the interval under consideration. This interval may be taken sufficiently small so that the signs of the differences, $y-Y$, for x_1+h_1 and x_2+h_2 are those of $h_1f'''(x_1)$ and $h_2f'''(x_2)$, respectively, for all values of h_1 and h_2 such that x_1+h_1 and x_2+h_2 fall within the interval. Since $f'''(x)$ does not change its sign in this interval and since h_1 is positive and h_2 is negative, the two parabolas lie on opposite sides of the curve within the given interval. Hence it is obvious that the intersections of the

*See Weber's, Lehrbuch der Algebra, 1st ed. I. p. 338.
†See Goursat-Hedrick's, Mathematical Analysis, vol. I, p. 93.

parabolas with the $X-axis$ lie on opposite sides of the root of $f(x)=0$, which falls within this interval.

The equation of the osculatory parabola at (x_1, y_1) is found to be $2Y=f''(x_1)x^2+2[f'(x_1)-x_1f''(x_1)]x+x^2_1f''(x_1)-2x_1f'(x_1)+2f(x_1)$. Its intercepts on the $X-axis$ are

$$x=x_1-f'(x_1)/f''(x_1)\pm\{[f'(x_1)/f''(x_1)]^2-2f(x_1)/f''(x_1)\}^{\frac{1}{2}}. \tag{1}$$

If $f(x_1)$ and $f''(x_1)$ have opposite signs, the roots of (1) are real and unequal. If these signs are the same, the roots are real and unequal if $2f(x_1)f''(x_1)$ is less than $[f'(x_1)]^2$. This condition can always be satisfied for it was assumed that $f'(x)$ and $f''(x)$ are different from zero and exist, making it possible for x_1 to be chosen near enough to the actual root so that $f(x_1)$, and therefore the product $2f(x_1)f''(x_1)$, can be made as small as desired. In particular it can be made smaller than $[f'(x_1)]^2$, a quantity not equal to zero.

The osculatory parabola from (x_1,y_1) may have only one intercept greater than x_1, in which case this intercept will be used as an approximation to the root of $f(x)=0$. In case it has two intercepts greater than x_1, the smaller of the two will be used. Likewise from (x_2, y_2) either the intercept smaller than x_2 or, if both are smaller, then the larger will be used. Observing the form of the solutions (1), it can be readily shown that the proper sign to choose for the radical is dependent upon the signs of $f'(x_1)$ and $f''(x_1)$. If $f'(x_1)$ and $f''(x_1)$ have the same signs use the positive sign for the radical, and if they have opposite signs use the negative sign. The same rule can be shown to apply for the intercepts of the osculatory parabola from (x_2, y_2).

In order to determine whether the approximation is less or greater than the root of $f(x)=0$, the function $f(x)$ is espanded into a Taylor's series about x_1 with a remainder after three terms. The expansion of the parabolic function (1) about x_1 consists of exactly three terms, each of which is equal to the corresponding terms of the expansion of $f(x)$. Hence the difference, $y-Y$, is equal to the remainder of the $f(x)$ expansion, which may be written in the form, $(x-x_1)^3f'''(a_1)/6$, where a_1 falls between x and x_1. Since the third derivative does not vanish in the interval $x_1\leqq x\leqq x_2$, the sign of $f'''(a_1)$ is known and furthermore it is the same as the sign of $y-Y$. If an expansion had been made about x_2 a similar result would have been obtained, differing only in that the sign of $y-Y$ is opposite that of the third derivative, since $(x-x_2)^3$ is negative.

When $f(x_1)$ is positive the parabola from (x_1 ,y_1) is below the curve if $y-Y$ is positive, and is above if it is negative. Hence the approximation from x_1 is less than or greater than the root depending upon

whether $f'''(x)$ is positive or negative, respectively. At the other end of the interval, $f(x_2)$ is negative and the parabola from (x_2, y_2) is above the curve if $y - Y$ is negative and is below if $y - Y$ is positive. Hence the approximation from x_2 is greater than or less than the root depending upon whether $f'''(x)$ is positive or negative, respectively. Likewise taking $f(x_1)$ negative, and hence $f(x_2)$ positive, it can be shown that the approximation from x_1 is less than or greater than the root, depending upon whether $f'''(x)$ is negative or positive, respectively. Also it can be shown that the approximation from x_2 is greater than or less than the root depending upon whether $f'''(x)$ is negative or positive, respectively. Since the first derivative is negative when $f(x_1)$ is positive and positive when $f(x_1)$ is negative, it will be seen that only two essentially different cases exist. First, if $f'(x)$ and $f'''(x)$ have opposite signs, and second, if they have the same signs.

In the first case the approximation from x_1 is less than the root and that from x_2 is greater than the root. Hence the maximum possible error in either approximation is numerically less than the absolute value of the difference between the two approximations. Furthermore it is obvious that at each step the approximations are better than the preceding ones.

In the second case the approximation from either end of the interval jumps over the root and hence, for each approximation, the maximum possible error is numerically less than the absolute value of the difference between the approximation and the value of x from which the approximation is taken. It can be shown that the approximation approaches the root as the point, from which the approximation is taken, approaches the root. Thus it is possible to work from either end of the interval or from both simultaneously. If the latter is done, the maximum possible error so obtained will generally be smaller than that by working from one end alone.

In applying the method described, the following steps may be found useful.

1. Determine an interval, $x_1 \leqq x \leqq x_2$, within which the root lies and such that

   (a) the derivatives $f'(x), f''(x), f'''(x)$ do not vanish, and

   (b) if $f(x_i)$ and $f''(x_i)$ have the same sign such that $2f(x_i)f''(x_i) < [f'(x_i)]^2$, where x_i is the x coordinate of the point of contact of the osculatory parabola with the curve.

2. Choose an approximation formula as follows;
   (a) if $f'(x)$ and $f''(x)$ have like signs use $x=x_i-f'(x_i)/f''(x_i)+\{[f'(x_i)/f''(x_i)]^2-2f(x_i)/f''(x_i)\}^{\frac{1}{2}}$, and
   (b) if $f'(x)$ and $f''(x)$ have unlike signs use $x=x_i-f'(x_i)/f''(x_i)-\{[f'(x_i)/f''(x_i)]^2-2f(x_i)/f''(x_i)\}^{\frac{1}{2}}$.

3. Determine the points from which the approximations are to be taken as follows;
   (a) if $f'(x)$, $f'''(x)$ are opposite in signs the approximation will remain on the same side of the root from which it was taken and hence approximations must be taken from both ends of the interval, and
   (b) if $f'(x)$, $f'''(x)$ are the same in signs the approximation will jump over the root and hence an approximation from either one end or the other of the interval may be taken or from both.

The following problems are given to illustrate the method.

Example 1. Find the irrational of $f(x)=x^3-4x^2-8x+5=0$, between 0 and 1.

None of the derivatives $f'(x)$, $f''(x)$, $f'''(x)$ vanishes in the interval from 0 to 1. A graph indicates that the root is slightly greater than 0.5. Since $f'(x)$ and $f''(x)$ have the same sign from 0 to 1 the formula from $2(a)$ is used and the approximation will remain on the same side of the root from which it is taken since $f'(x)$ and $f'''(x)$ are opposite in sign. Using a computing machine the approximation from 0.5 is 0.51108382. This is probably correct to several decimal places so it is safe to assume that 0.52 is beyond the root. The approximation from 0.52 is 0.51108399. Either approximation gives the root with an error numerically less than the quantity 0.00000017. Using a 6-place logarithm table the approximation from both 0.5 and 0.52 is 0.51108, which is sufficiently accurate for most practical purposes.

Example 2. Find the irrational root of $f(x)=x^3-2x-5=0$, between 2 and 3.

The approximation formula from $2(a)$ is again used but since $f'(x)$ and $f'''(x)$ have like signs from 2 to 3, the approximation jumps over the root. A graph indicates that the root is a little less than 2.1. Using a computing machine the approximation from 2.1 is 2.09455147. Using 2.09 as the other end of the interval, the approximation is 2.09455149 and hence the numerical value of the error in either approximation is less than 0.00000002. Using a 6-place table of logarithms the approximation from either end is 2.094551.

A Certain Geometrical Calculus by *Van Der Waerden* in Connection with the Elliptic Cubic

By Arnold Emch
University of Illonois

1. In his "Einführung in die algebraische Geometrie" (1939), pp. 93-94, *Van Der Waerden* establishes a formal geometrical calculus, which is a partial equivalent of the properties of the arguments in the representation of general plane cubics by elliptic functions. By it he is able to prove a number of propositions on the cubic, in particular the theorem that the cross-ratio of the four tangents from a point of the cubic to the cubic is sonstant. But the calculus fails when problems involving the periods of elliptic functions are concerned. It is the purpose of this paper to cover *Van Der Waerden's* work directly by the application of elliptic functions and to show where the proofs are incomplete.

2. Assume two fixed points R, Q, on an elliptic cubic C_3. Through R draw any line a, cutting C_3 in A_1, A_2. Join these with Q and obtain the intersections B_1, B_2. It is to be proved that *the join* B_1, B_2 *passes through a fixed point* S *on* C_3. By elliptic arguments indicated by small letters corresponding to the points R, Q, S, A_1, etc., there is

$$r+a_1+a_2\equiv 0 \text{ (period } 2m_1\omega_1+2m_2\omega_2)$$

$$q+a_1+b_1\equiv 0\,,$$

$$q+a_2+b_2\equiv 0\,.$$

From this $\quad 2q+a_1+a_2+b_1+b_2\equiv 0$, or

$$2q-r+b_1+b_2 \equiv 0\,.$$

But $\quad s+b_1+b_2 \equiv 0$, hence

(1) $\quad s=-r+2q \equiv 0$,

so that S is a fixed point, q.e.d.

3. *An application of Noethers' Theorem.*

Suppose that two distinct curves C_m and C'_m cut C_3 in the same $3m$ points. Then any other curve through these may be written as

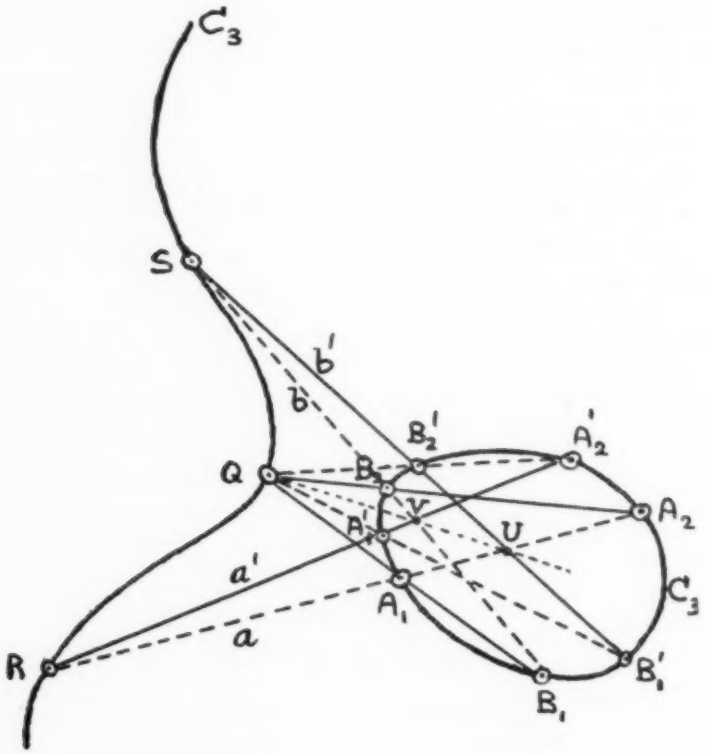
$A \cdot C_m + A'C'_m = 0$, including C_m amd C'_m themselves. Since C'_m passes through the full intersection of C_3 and C_m we have, for $m \geqq 3$,

$$C'_m = k \cdot C_m + C_3 \cdot C_{m-3} . \tag{2}$$

Form the pencil

$$C'_m + \lambda C_m = (k+\lambda)\ C_m + C_3 \cdot C_{m-3} . \tag{3}$$

For $\lambda = -k$ this reduces to $C_3 \cdot C_{m-3}$. Thus *among the curves of the pencil is a degenerate curve which is compounded of* C_3 *and a* C_{m-3}. C'_m and C_m intersect in m^2 points, among which are the $3m$ points on C_3. Hence *the remaining* $m(m-3)$ *points do not lie on* C_3, *but must lie on a* C_{m-3}.



4. Now apply this to the configuration of the figure. The quartics $C_4 = a \times b \times A_1'B_1' \times A_2'B_2'$ and $C_4' = a'b' \times A_1B_1 \times A_2B_2$ intersect in 16 points of which 12, i.e., R, S, Q^*, Q, A_1, B_1, A_1', B_1', A_2, A_2', B_2, B_2' lie on C_3. Hence the *remaining four*, Q, Q, V, V lie on a $C_{4-3} = C_1$, a

*Q counted twice.

straight line. Now let a', b' fixed while a varies. The series of points (A_1) is projected upon a series of points (V) on b' from R. From Q project (V) into a series (V) on a', which is obviously projective to (V). Thus the *Theorem*: *When a describes a pencil, b describes a projective pencil. Among the lines a are the four tangents from R to* C_3. *From the way in which* B_1, B_2 *were obtained from* A_1, A_2, *it is clear that the corresponding b's are the four tangents from S. Hence the cross-ratio of the four tangents from R is equal to that of the four tangents from S.*

Instead of choosing R and Q as given, choose R and S. Then according to (1) we find

$$q=\tfrac{1}{2}(r+s)+\tfrac{1}{2}\text{ period}\,. \tag{4}$$

From this follows the well known (Salmon) *Theorem*: *The cross-ratio of the four tangents from a point of an elliptic cubic to this cubic is constant (of course taken in the proper order).* Another new proof of this theorem by the author, based upon the involutorial quadratic transformation leaving the cubic invariant, will appear in an early number of the *Monats hefte für Mathematik und Physik.*

5. From (4) it is apparent that for a definite arbitrary choice of R and S, there are four associated Q's

$$Q_1=\tfrac{1}{2}(r+s)\,,$$

$$Q_2=\tfrac{1}{2}(r+s)+\omega_1\,,$$

$$Q_3=\tfrac{1}{2}(r+s)+\omega_2\,,$$

$$Q_4=\tfrac{1}{2}(r+s)+\omega_1+\omega_2\,.$$

This is what the geometrical calculus cannot establish. Thus *Van Der Waerden* simply says: An adequate Q may be found. We proceed to draw the tangents at each of these points and find their tangentials $T_i(t_i)$, $i=1, 2, 3, 4$. There is for

$$Q_1\quad (r+s+t_1\equiv 0)\,,$$

$$Q_2\quad (r+s+2\omega_1+t_2\equiv 0)\,,$$

$$Q_3\quad (r+s+2\omega_2+t_3\equiv 0)\,,$$

$$Q_4\quad (r+s+2\omega_1+2\omega_2+t_4\equiv 0)\,,$$

from which follows $t_1=t_2=t_3=t_4\equiv -(r+s)$, that the tangentials coincide and that $Q_1\,Q_2\,Q_3\,Q_4$ is a Steinerian quadruple. Each of the four Q's induces a projectivity π_1 between the two pencils of tangents from

R and S. According to the four-group G_4 we have the four projectivities with the same cross-ratio

$$\pi_1 = (r_1r_2r_3r_4)\overline{\wedge}(s_1s_2s_3s_4) ,$$

$$\pi_2 = (r_1r_2r_3r_4)\overline{\wedge}(s_4s_3s_2s_1) ,$$

$$\pi_3 = (r_1r_2r_3r_4)\overline{\wedge}(s_3s_4s_1s_2) ,$$

$$\pi_4 = (r_1r_2r_3r_4)\overline{\wedge}(s_2s_1s_4s_3) ,$$

Here the r_1 and s_1 denote tangents. The projectivity π_i generates a conic K_i on R and S. It is easily established that K_i passes through the four points of tangency of the tangents from Q_i to C_3. The result may be stated as

Theorem: *The four projectivities π_i generate four conics K_i on R and S. The four r_i cut the four s_i in 16 points which by fours lie on the four conics K_i. K_i cuts C_3, outside of R and S, in four points which are the points of tangency of the tangents from Q_i to C_3. Thus the K_i cut C_3 in four Steinerian quadruples.*

Each couple, RQ, QS, SR gives rise to four conics K_i each cutting C_3 in a Steinerian quadruple. Thus there are 12 such quadruples.

A scientist should be the humblest of men. He soon learns that he dwells not in a world of reality but in a make-believe world. Reality is forever beyond him. He moves slowly toward it, but it ever eluded him. As Max Planck has said, we are compelled to contemplate the nature of things through spectacles of whose optical properties we are entirely ignorant and whose elements of design must ever remain unknown to us.

But though science is by its very nature an artificially built-up and make-believe world, this does not mean that it does not direct our gaze toward truth and reality, and of course it does not mean that mathematics is not the most potent guide to that end.

—From Charles S. Slichter's "*Science in a Tavern*", Published by the University of Wisconsin Press.

A Non-Rotational Operator of Third Order Periodicity

By A. B. SOBLE
Philadelphia, Pa.

The simplest non-rotational operators of second order periodicity are $1/z$ and $1-z$, while the simplest non-rotational operators of *third* order periodicity are $1/(1-z)$ and its inverse $(z-1)/z$. Although the order-periodicity of these operators has been known from earliest times, it seems that the order-periodicity of the operator presented in this article has hitherto not been recognized.

Let the operator $i \log_e \cot \frac{1}{2}$ be represented by the symbol C, the inverse by C^{-1}. Let $C^{n+1} = C(C^n)$.

$C(x) = i \log \cot \frac{1}{2}x = i \log \tan (\frac{1}{2}\pi - \frac{1}{2}x) = i \log \tan [\frac{1}{4}\pi + \frac{1}{2}(\frac{1}{2}\pi - x)]$.

Hence

$$C(x) = i\, gd^{-1}(\tfrac{1}{2}\pi - x)^*. \tag{I}$$

Now $\exp \mathrm{gd}^{-1} x = [\exp \mathrm{gd}^{-1}(\frac{1}{2}\pi - x) + 1]/[\exp \mathrm{gd}^{-1}(\frac{1}{2}\pi - x) - 1]$†

$= \coth [\frac{1}{2}\mathrm{gd}^{-1}(\frac{1}{2}\pi - x)]$. Hence $\mathrm{gd}^{-1} x = \log \coth [\frac{1}{2}\mathrm{gd}^{-1}(\frac{1}{2}\pi - x)]$

$= \log \{i \cot [\frac{1}{2} i\, \mathrm{gd}^{-1}(\frac{1}{2}\pi - x)]\} =$, by (1), $\log [i \cot \frac{1}{2}C(x)]$

$= \log i + \log \cot \frac{1}{2}C(x)$. Thus $i\, \mathrm{gd}^{-1} x = i \log i + i \log \cot \frac{1}{2}C(x)$

$=$, by definition, $i \log i + C^2(x) = -\frac{1}{2}\pi + C^2(x)$. Therefore

$C^2(x) = \frac{1}{2}\pi + i\, \mathrm{gd}^{-1} x$. Now $i\, \mathrm{gd}^{-1} x = \mathrm{gd}\,(ix)$†. Hence

$$C^2(x) = \tfrac{1}{2}\pi + \mathrm{gd}(ix). \tag{II}$$

Let $\mathrm{gd}^{-1} w = z$. Therefore $z = \log \tan (\frac{1}{4}\pi + \frac{1}{2}w)^*$. Hence $e^z = \tan (\frac{1}{4}\pi + \frac{1}{2}w) = \cot (\frac{1}{4}\pi - \frac{1}{2}w) = \cot \frac{1}{2}(\frac{1}{2}\pi - w)$. Thus 2 arc cot $e^z = \frac{1}{2}\pi - w = \frac{1}{2}\pi - \mathrm{gd}\, z$. Let $z = -iy$. Therefore 2 arc cot $e^{-iy} = \frac{1}{2}\pi - \mathrm{gd}(-iy)$. Now the gudermannian is an odd function‡. Hence 2 arc cot $e^{-iy} = \frac{1}{2}\pi + \mathrm{gd}(iy) =$, by (II), $C^2(y)$. Now let $C(u) = v$. Hence $u = C^{-1}(v)$ and, by definition, $i \log \cot \frac{1}{2}u = v$. Thus $C^{-1}(v) = u = 2$ arc cot e^{-iv}. Therefore

$$C^2 = C^{-1}$$

*Vid. almost any book on college trigonometry or hyperbolic functions.

†Günther, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*, 1881, p. 115.

‡Gudermann, (Crelle's JOURNAL f. d. REINE u. ANGEW. MATH., vierter Band, 1829), *Ueber die Potenzial-Functionen*, p. 290.

Hence C is an operator of third order periodicity.

We show now that C is not a rotation through $2\pi/3$.

$C(\pi/3 = i \log \cot \pi/6 = i \log \sqrt{3} = \frac{1}{2}\ i \log 3$. Now amp $\pi/3 = 0$, while amp $[\frac{1}{2} i \log 3] = \pi/2$. Hence C is not a rotation.

Therefore the inverse of C, $2 \tan^{-1} \exp i$, is also a non-rotational operator of third order periodicity.

Moreover, if R be ANY operator and R^{-1} its inverse, then the operators

$$D \equiv RCR^{-1} \quad \text{and} \quad d \equiv RC^{-1} R^{-1}$$

are also operators of third order periodicity, in general non-rotational. This follows at once on simplifying DDD and ddd.

If $R = \frac{1}{2}$, then $D = i/2 \log \cot$, which is even neater than C.

If $R^{-1} = i \log$, then $D(x)$ reduces to $i(1+x)/(1-x)$. This follows from the exponential definitions of the sine and cosine. Now it is easily verified that $E[i(1+x)/(1-x)] = \omega E(x)$, where $E(z) \equiv [\ +(\omega^2 - i\omega)]/[z+(\omega - i\omega^2)]$ and $\omega^3 = 1$. Therefore $ED = \omega E$. Thus $D = E^{-1} \omega E$. Hence, by the definition of D, $R\ CR^{-1} = E^{-1}\ \omega E$. Therefore $C = R^{-1} E^{-1} \omega E R$. Hence

$$C = G^{-1}\, \omega\, G\ ,$$

where $$G(z) \equiv [1+(\omega^2 - i\omega)e^{iz}]/[1+(\omega - i\omega^2)e^{iz}]\ .$$

Operators of n^{th} order-periodicity may be constructed from the periodicity of $F \rho F^{-1}$, where F is any operator and $\rho^n = 1$. A simple illustration is given by cosh cos^{-1}, formed from cos i cos^{-1}. That cosh cos^{-1} is not a rotation may be seen by choosing $\frac{1}{2}\sqrt{2}$ as argument.

The periodicity of C may also be proved simply, without the use of the gudermannian, by twice using the identity

$$C(i \log Z) \equiv i \log [(i+iZ)/(1-Z)]$$

as a formula, substituting for Z first cot $\frac{1}{2}z$ and then $(i+i \cot \frac{1}{2}z)/(1 - \cot \frac{1}{2}z)$.

On Brocard's Ellipse

By VINCENZO CAVALLARO
Cefalu, Sicily

1. In this note we give some relations among the constants of Brocard's ellipse. ABC represents the fundamental triangle; O and R the center and radius of the circumscribed circle; K Lemoine's point; Ω,Ω' Brocard's points; V Brocard's angle; σ the radius of Brocard's circle; ρ_1 and ρ_2 the radii of the first and second Lemoine circle; 2α and 2β the axes of Brocard's ellipse (of center ω and foci Ω, Ω'). It is known that

$$(1)\qquad \sigma=\frac{R}{2}\sqrt{1-3\tan^2 V}\ ;\ \rho_1=\frac{R}{2}\sec V;\ \rho_2=R\tan V$$

$$(2)\qquad 2\alpha=2R\sin V;\ 2\beta=4R\sin^2 V$$

In consequence of (1) we have

$$(3)\qquad \rho_1^2=\rho_2^2+\sigma^2;\ R^2=3\rho_1^2+\sigma^2=3\rho_2^2+4\sigma^2=4\rho_1^2-\rho_2^2$$

$$(4)\qquad R=2\rho_1\cos V=\rho_2\cot V;\ 1-4\sin^2 V=\frac{\sigma^2}{\rho_1^2};\ \sin V=\frac{\rho_2}{2\rho_1}$$

Then

$$(5)\qquad 2\alpha=2\rho_1\sin 2V=2\rho_2\cos V$$

$$2\beta=4\rho_1\sin V\sin 2V=2\rho_2\sin 2V$$

and

$$(6)\qquad \sin V=\frac{\rho_2}{2\rho_1}=\frac{\alpha}{R};\ \cos V=\frac{\alpha}{\rho_2}=\frac{R}{2\rho_1};\ \sin^2 V=\frac{\alpha}{\rho_1}=\frac{\beta}{\rho_2}$$

$$(7)\qquad \frac{2\alpha}{2\beta}=\frac{\rho_1}{\rho_2}=\frac{R}{2\alpha}=\frac{1}{2\sin V};\ (2\alpha)^2=2\beta\cdot R$$

$$2\alpha=R\frac{\rho_2}{\rho_1}\ ;\ 2\beta=R\frac{\rho_2^2}{\rho_1^2};\ 2\alpha\cdot 2\beta=R^2\frac{\rho_2^3}{\rho_1^3}$$

The Brocard quadrilateral $O\Omega K\Omega'$ is inscribed in Brocard's circle of diameter $OK=2\sigma$; OK is perpendicular to $\Omega\Omega'$ at the point ω;

$$\underline{|\Omega O\omega}=\underline{|\Omega' O\omega}=V,$$

so that $O\Omega = O\Omega'$; $K\Omega = K\Omega'$; $\Omega\omega = \Omega'\omega$

We have then $O\Omega = 2\sigma\omega V$, and, in virture of (1) ,(4):

$$O\Omega^2 = R^2(1-4\sin^2 V);\ O\Omega = \frac{R\sigma}{\rho_1} \tag{8}$$

Since, (6), $\sin V = \dfrac{\alpha}{R}$, we have

$$O\Omega^2 = R^2 - 4\alpha^2\ ;\ R^2 = O\Omega^2 + (2\alpha)^2 \tag{9}$$

From the quadrilateral $O\Omega K\Omega'$, (5) and (7) we find $\Omega\Omega' = 2\sigma\sin V$.

$$\frac{\Omega\Omega'}{2\sigma} = \frac{\sigma}{\rho_1}\ ;\ \frac{\Omega\Omega'}{2\beta} = \frac{\sigma}{\rho_2} \tag{10}$$

$$\Omega\Omega' = \frac{R\sigma\rho_2}{\rho_1^2} \tag{11}$$

In the ellipse $\Omega\Omega'^2 = 4\Omega\omega^2 = (2\alpha)^2 - (2\beta)^2$

hence from (7):

$$\Omega\Omega'^2 = R^2\left[\left(\frac{\rho_2}{\rho_1}\right)^2 - \left(\frac{\rho_2}{\rho_1}\right)^4\right] \tag{12}$$

From the quadrilateral $O\Omega K\Omega'$, $\Omega\Omega'^2 = 4\Omega\omega^2 = 4O\Omega^2\sin^2 V$. If O_g and R_g are the center and the radius of the nine-point circle, it is known that $R = 2R_g$and, from the Weill-Aiyar theorem

$$O\Omega^2 = 2R\cdot O_g\omega \tag{13}$$

Then, from (2), we have

$$\Omega\Omega' = 2\sqrt{O_g\omega\cdot\beta} \tag{14}$$

Now from (2), (8), and (13)

$$O_g\omega = \frac{\mathrm{R}}{2} - \beta \tag{15}$$

and from (4)

$$O_g\omega = \frac{R\sigma^2}{2\rho_1^2} = \frac{R_g\sigma^2}{\rho_1^2} \tag{16}$$

From (7),

$$\frac{R}{2}=\alpha\frac{\rho_1}{\rho_2}=\beta\frac{\rho_1^2}{\rho_2^2} \quad \text{hence}$$

$$O_g\omega=\frac{\alpha\sigma^2}{\rho_1\rho_2}=\frac{\beta\sigma^2}{\rho_2^2} \tag{17}$$

In the quadrilateral $O\Omega K\Omega'$: $O\omega^2=O\Omega^2\omega^2V$. Hence from (9),

$$O\omega^2=R^2(1-4\sin^2 V)\cos^2 V=R^2(1-\sin^2 V-\sin^2 2V) \tag{18}$$

Since $\cos V=\dfrac{R}{2\rho_1}$, (6), we have

$$O\omega=\frac{R^2\sigma}{2\rho_1^2} \tag{19}$$

and, from (16)

$$\frac{O\omega}{O_g\omega}=\frac{R}{\sigma} \tag{20}$$

Also (4), (6), (18) give

$$O\omega=\frac{R\sigma\alpha}{\rho_1\rho_2} \tag{21}$$

2. *Powers.* If $P(\Omega,O)$, $P'(\Omega',O)$ are the powers of the foci Ω,Ω' with respect to the circumscribed circle $O(R)$, then $P(\Omega,O)=P(\Omega',O)=R^2-O\Omega^2$, then from (8), (4), we have

$$P(\Omega,O)=R^2-R^2(1-4\sin^2 V)=(2R\sin V)^2=(2\alpha)^2 \tag{22}$$

If $P(\omega,O)$ is the power of the center ω with respect to $O(R)$, we have $P(\omega,O)=R^2-O\omega^2$, and, from (18)

$$P(\omega,O)=R^2[\sin^2 V+\sin^2 2V] \tag{23}$$

Also from (21), (6) we have

$$P(\omega,O)=R^2\left[1-\frac{\sigma^2\alpha^2}{\rho_1^2\rho_2^2}\right]=R^2\left(1-\frac{\sigma^2\beta^2}{\rho_2^4}\right) \tag{24}$$

$P(\omega,O_g)$ is the power of the center ω with respect to the nine-point circle

$$O_g(R_g)=O_g\left(\frac{R}{2}\right)$$

$$P(\omega,O_g)=\frac{R^2}{4}-O_g\omega^2$$

From (16)

(25) $$P(\omega,O_g)=\frac{R^2}{4}\left(1-\frac{\sigma^4}{\rho_1{}^4}\right)$$

and from (7), (15)

(26) $$P(\omega,O_g)=\frac{(2\alpha)^2}{2}-\frac{(2\beta)^2}{4}\ .$$

Remark.

It is easy to prove, for preceding relations, that

$$K\Omega=\sigma\frac{\rho_2}{\rho_1};\quad 2K\omega=\sigma\frac{\rho_2{}^2}{\rho_1{}^2}$$

$$\frac{O\Omega}{\Omega\Omega'}=\frac{\rho_1}{\rho_2}=\frac{2\alpha}{2\beta}=\frac{K\Omega}{2K\omega}$$

$$O\Omega^2=\Omega\Omega'^2+(2O_g\omega)^2\ ;\ R^2=\Omega\Omega'^2+(2\alpha)^2+(2O_g\omega)^2$$

$$(2O_g\omega)^2=O\Omega^2+(2\beta)^2-(2\alpha)^2;\quad \frac{1}{\alpha^2}=\frac{1}{R^2}+\frac{1}{\rho_2{}^2}$$

Here is the situation as I see it: Mathematics is under fire because it is hard and because, being hard, pupils do not get as much out of it as they should for the time and energy used. One chief reason why it is hard is that we teach it to our students before they are mentally ready to cope with it adequately. One reason why we teach it to them too early is that it has always been considered so important that it has been felt that they must get it, and get it as early as it is possible for them to receive it. Hence we are arriving at the argument (which is more logical than it sounds when put in this way) that we must throw out mathematics because it is so very important.

—From an editorial published by Chas. H. Sisam, Colorado College, in *National Mathematics Magazine*, December, 1934.

The Rectifying Curve

By Roy A. Mather
Leetsdale, Pa.

The purpose of this paper is to give the derivation of a curve for rectifying circular arcs, and to make some applications.

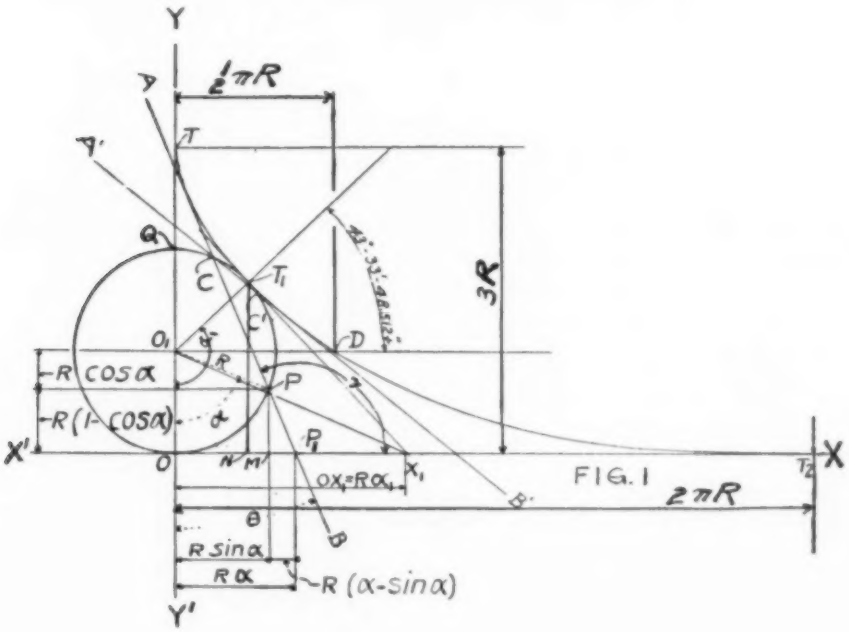


FIG. 1

In Figure 1, the circle of radius R equal to O_1O and centered at O_1 on the axis Y, is called the base circle. The straight line AB is attached to the base circle at P, called the pivot point. Let* $@ = \angle OO_1P$, $\theta = \angle MPB$, and $\gamma = \angle XP_1P$. Let AB in its initial position coincide with OY and P with O, where $@ = 0°$, $\theta = 0°$, and $\gamma = 90°$. When the circle is turned counter-clockwise about O_1, let AB rotate on P so its $X-$intercept shall always be equal to $R@$. The locus of the intersections of successive positions of AB as $@$ increases from 0° to 360° is the envelope $TT_1\,DT_2$ of AB and is called the rectifying curve.

*Note to the Editor: @ in the text = α in the diagram, Fig. 1.

The equation of AB, having $X-\text{intercept}=OP_1$ and slope

$$m=\tan\gamma=-\cot\theta=-\frac{MP}{MP_1}=-\frac{(1-\cos @)}{@-\sin @}\ ,\ \text{is}$$

$$x+\frac{@-\sin @}{1-\cos @}\,y=R@. \tag{1}$$

To find the envelope of AB, differentiate in (1) with respect to @, and obtain

$$y=\frac{(1-\cos @)^2}{2(1-\cos @)-@\sin @}\,R\,. \tag{2}$$

Eliminating y in (1) by (2),

$$x=\left[@-\frac{(1-\cos @)\,(@-\sin @)}{2(1-\cos @)-@\sin @}\right]R\,. \tag{3}$$

Equations (2) and (3) are the rectangular equations of the rectifying curve in the variable parameter @.

In finding the $Y-$intercept T of the rectifying curve, where $@=0°$ and $\gamma=90°$ the steps are shortened by first substituting in (2) and (3),

$$\sin @={}^1\!/_6\,@^3+\cdots$$

and

$$1-\cos @={}^1\!/_2\,@^2-{}^1\!/_{24}\,@^4+\cdots$$

which gives

$$x=\left[@-\frac{{}^1\!/_{12}\,@+\cdots}{{}^1\!/_{12}\quad+\cdots}\right]_{@=0°}^{R=0}$$

and

$$y=\left[\frac{{}^1\!/_4-\dfrac{@^2}{24}+\cdots}{{}^1\!/_{12}-\dfrac{@^2}{180}+\cdots}\right]_{@=0°}^{R=3R}$$

for the coordinates of T. Since AB coincides with OY the rectifying curve is tangent to OY.

The $X-$intercept T_2 is found by letting $@=360°$. The indeterminate forms derived are evaluated by differentiating the numerators and denominators separately in (2) and (3). Thus for T_2,

$$x=2\pi R \text{ and } y=0.$$

Also, when $@=360°$, $\gamma=180°$, and AB coincides with OX. Hence the rectifying curve is tangent to OX. The rectifying curve is thus found to be a finite curve.

As $@$ increases from 0°, the pivot point P moves up counterclockwise while the point of tangency of AB with the rectifying curve moves down from T, and will coincide with P at T_1 of $@=@_1$. Coincidence of P and T_1 occurs when

$$y=\frac{(1-\cos @_1)^2}{2(1-\cos @_1)-@_1 \sin @_1}R=NT_1=(1-\cos @_1)R,$$

or, when $$\frac{(1-\cos @_1)^2}{2(1-\cos @_1)-@_1 \sin @_1}=1-\cos @_1$$

which reduces to

$$@_1=\tan {}^1\!/\!_2\, @_1. \tag{4}$$

Now the $X-$intercept of the tangent to the circle at $T_1=OX_1$, and, Fig. 1,

$$OX_1=R \tan {}^1\!/\!_2\, @_1=R@_1.$$

The tangent to the rectifying curve at T_1 and the tangent to the circle at T_1, both passing through the point OX, must coincide. Hence the rectifying curve is tangent to the base circle at T_1, and their normals at T_1 coincide. The equation (4) where $@_1$ is incommensurable has the solution $@_1$, which is the measure in radians of an arc of

$$@_1=133°\ 33'\ 48.512''+.$$

By making $@=\pi$ in (2) and (3),

$$x={}^1\!/\!_2\pi R \text{ and } y=R.$$

Hence the rectifying curve crosses the extended horizontal diameter of the base circle at D.

The rectifying curve has the property that a straight line AB through the pivot point P, tangent to the rectifying curve, and prolonged to cut OX, rectifies the arc OP into the straight line OP_1. Conversely, a straight line AB through P_1, tangent to the rectifying curve, and cutting the circumference of the base circle at P, lays off the straight line OP_1 into an equal distance, arc OP, on the base circle. The tangent line AB touches the curve above T_1 when $@<@_1$ and below T_1 when $@>@_1$.

In rectifying an arc, $@>270°$, where the intersection of the corresponding tangent with OX is not sharp, rectify the arc of $@-270°$

and add to it $3(O_1D)$, the rectified arc of 270°. In rectifying an arc @ near $@_1$, where the pivot point is close to the tangent point, rectify the half arc and double the result.

The rectifying curve can be used for trisecting an angle, for dividing an angle into any given ratio, for squaring a circle, and for finding a rectangle equivalent in area to the segment of a circle.

To do these constructions graphically the rectifying curve may be laid out to a convenient scale adopted for the base circle, preferably on a piece of translucent celluloid. Shape one edge to the curve and axes $X'X$ and $Y'Y$. Inscribe on both sides of it line O_1D extended, and the normals through T and T_1. By the aid of these lines this curve so constructed can readily be set on the diagram of the base circle so that it will coincide with and take the place of the rectifying curve. A straight edge laid to the pivot point will find its tangent point against this curve.

To rectify an arc OP on the base circle in Fig. 2, draw a straight line through P, touching the rectifying curve, and cutting OX at P_1.

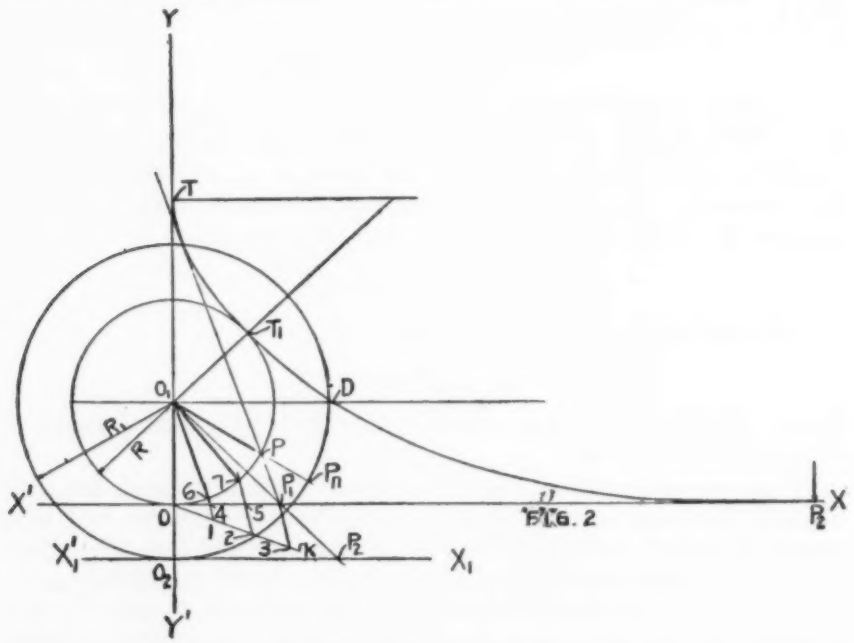


FIG. 2

Then OP_1 is the rectified arc OP. To rectify an arc O_2P_n on a concentric circle of radius R_1, draw the line O_1P_n intersecting the base circle at P and rectify arc OP into the line OP_1. Draw O_1P_1 extended to P_2 on O_2X_1. Then O_2P_2 is the rectified arc O_2P_n.

To trisect an angle OO_1P, draw a random line OK as in Fig. 2, and lay off on it three equal parts 01, 12, and 23. Having rectified the arc OP into the line OP_1, connect 3 and P_1 by a straight line and draw lines parallel thereto through 1 and 2 to intersect OP_1 at 4 and 5. Now draw lines through 4 and 5, touching the rectifying curve, and cutting the base circle in 6 and 7. Thus the arc OP is trisected by 6 and 7, for $04 = \text{arc } 06$, $45 = \text{arc } 67$, and $5P = \text{arc } 7P$. Hence the angles 00_16, 60_17, and 70_1P are equal. An angle may be divided into any given ratio by laying off that ratio on OK, Fig. 2, and proceeding as before.

To square the base circle, rectify OT_1Q, Fig. 3, into $OP_1 = \pi R$. The rectangle $OP_1NO_1 = \pi R^2$. Now construct OP as the mean pro-

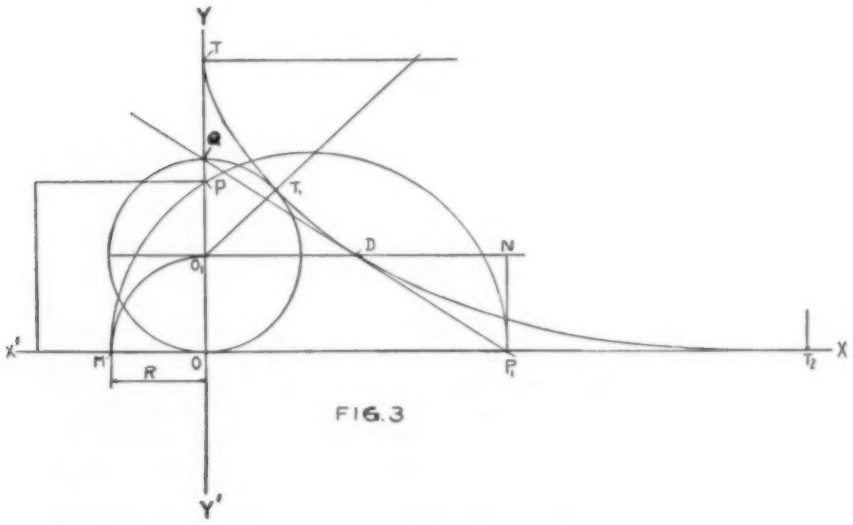


FIG. 3

portional between OP_1 and $OM = 00_1$. Then $OP^2 = OP_1 \cdot OM = \pi R^2$. Any other circle may be similarly squared, see Figs. 2 and 3.

To reduce the area of a segment of a circle to an equivalent rectangle, Fig. 4, rectify arc OP into OP_1. Then area of sector

$$OO_1P = \tfrac{1}{2}R \cdot OP_1 = \text{rectangle } OP_1AB;$$

area of triangle $OO_1P = \tfrac{1}{2}R \cdot OD = \text{rectangle } ODCB;$

hence the area of segment

$$OP = {}^1/_2(OP_1 - OD) = \text{rectangle } CDP_1A.$$

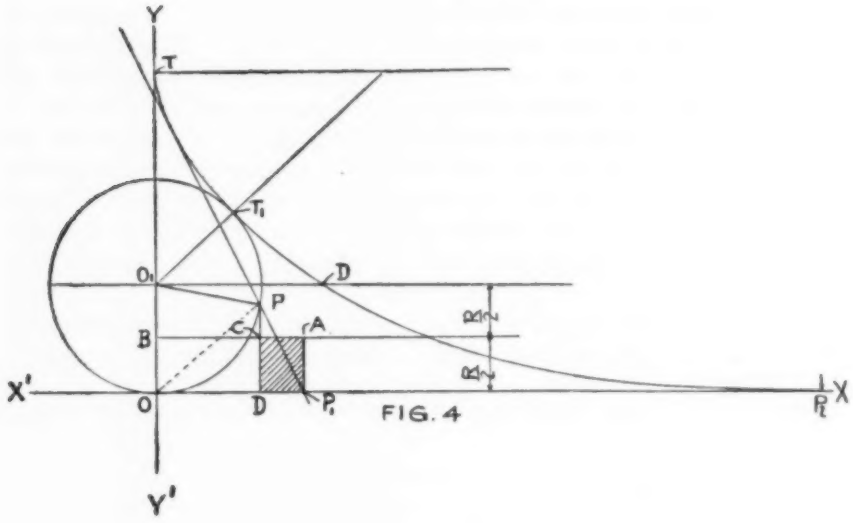


FIG. 4

A sector of any other circle may be similarly reduced to an equivalent rectangle. See Figs. 2 and 4.

In teaching mathematics, then, hold the student to thorough and rigid processes of reasoning, to accurate and precise thought and expression, to clear-cut methods which will not tolerate inaccuracy or slovenliness; in a word, let him be trained so that the man is master of his mind. Again, let the student be told of the many and varied applications of the science; let him know that it is a necessity to business men, surveyors, architects, cival engineers, chemists, physicists, navigators, astronomers. Teach all these with emphasis; a subject does not lose its educational value because of it practical bearings, nor is a theorem shorn of its beauty by reason of its usefulness. Finally, whenever opportunity offers, direct attention to those truths which yield real pleasure and pure enjoyment.

—From Alfred Hume's "*Backbone in a Curriculum*".

Remarks on Temple's Trigonometric and Hyperbolic Analogies*

By H. A. SIMMONS

§1. *Introduction.* Recently we derived independently the main results which Professor V. B. Temple reported last March in connection with the differential equations

$$\frac{d^n y}{dx^n}+y=0, \tag{1}$$

$$\frac{d^n y}{dx^n}-y=0; \tag{1.1}$$

and we observed relative to his "circular- and hyperbolic-analogies" a few facts of interest that he did not report. It is the purpose of this note to mention these additional facts and to set forth alternative methods of proving two important results of his article. We give these alternative methods first. We use his notations.

§2. *Avoidance of use of a primitive nth root of unity.* Relative to (1), the characteristic equation $M^n+1=0$ has n distinct roots $e_i (i=1,\cdots, n)$. On page 266, Temple says: "If ω is a primitive nth root of unity, we may suppose the e_i's to be arranged so that

$$e_i=e_1\omega^i, \quad i>1." \tag{2}$$

Were this true, the case $i=n$ would read: $e_n=e_1$ (contradiction). After this mistake, it was necessary to make another one in order to pass from equation (21) to (24) of his article. However, one can employ a primitive nth root of unity and pass through (22) and (23) correctly by substituting for (2) the relations

$$e_i=e_1\omega^{i-1} \quad (i=1,\cdots, n).$$

Without using a primitive nth root of unity, we may evaluate the expression (cf. his equation (22))

$$1-\sum_{k=2}^{n} e_i^{k-1}\, e_j^{n-k+1} \tag{2.1}$$

*Cf. V. B. Temple, this Magazine, March, 1939, pp. 263-271.

as follows. Writing out the expression (2.1), we obtain

$$1-(e_ie_j^{n-1}+e_i^2e_j^{n-2}+\cdots+e_i^{n-1}e_j). \tag{2.2}$$

If $i=j$, each product in (2.2) equals -1, and so this entire expression equals n. If $i\neq j$, division of (2.2) by $-e_i^n(=1)$ gives the expression

$$1+\frac{e_j}{e_i}+\left(\frac{e_j}{e_i}\right)^2+\cdots+\left(\frac{e_j}{e_i}\right)^{n-1},$$

which equals zero since the quotient of two nth roots of -1, $n>1$, is an nth root $\neq 1$ of 1 (i. e.,

$$m=\left(\frac{e_j}{e_i}\right)$$

satisfies the equation $1+m+m^2+\cdots+m^{n-1}=0$, which one obtains on dividing (m^n-1) by $(m-1)$).

In connection with "hyperbolic analogies", Temple finds it necessary to evaluate the expression (cf. his equation (38))

$$1+\sum_{k=2}^{n} e_i^{k-1}e_j^{n-k+1}, \tag{2.3}$$

and he does this elegantly by means of a primitive root ω of unity such that $e_i=\omega^i$. Again, we may avoid the use of a primitive root. Written out, the expression (2.3) is

$$1+e_ie_j^{n-1}+e_i^2e_j^{n-2}+\cdots+e_i^{n-1}e_j. \tag{2.4}$$

If $i=j$, each product in (2.4) equals 1, and so the sum in (2.4) is n. If $i\neq j$, division of (2.4) by $e_i^n(=1)$ gives the equal expression

$$1+\frac{e_j}{e_i}+\left(\frac{e_j}{e_i}\right)^2+\cdots+\left(\frac{e_j}{e_i}\right)^{n-1},$$

which equals zero since any quotient of two distinct roots of $m^n-1=0$ is a root of $1+m+m^2+\cdots+m^{n-1}=0$.

§3. *Avoidance of lengthy computation in deriving addition formulas**. If $f(x)$ is a solution of a linear differential equation with constant coefficients, then $f(x+y)$, y an arbitrary constant, is also a solution. Hence $u_1(x+y)$ must be a solution of (1). To determine its form,

*The familar method employed in this section was recalled to the author by Professor D. R. Curtiss.

which is necessarily linear in n linearly independent solutions of (1), we assume that

$$u_1(x+y)=u_1(x)a_1(y)+u_2(x)a_2(y)+\cdots+u_n(x)a_n(y). \tag{3}$$

Setting $x=0$ and using the boundary conditions, (26) and (27) of Temple's article, one finds that $a_1(y)=u_1(y)$. Differentiating equation (3) partially with respect to x and applying the boundary conditions again, one finds that $-a_n(y)=(-1)^n u_2(y)$ so that

$$a_n(y)=-(-1)^n u_2(y).$$

Then one further differentiation and application of the boundary conditions shows that

$$a_{n-1}(y)=-(-1)^{n-1}u_3(y);$$

etc., until one obtains the result (given in equation (24) of Temple's article)

$$u_1(x+y)=u_1(x)u_1(y)-(-1)^n\sum_{k=2}^{n}u_k(x)u_{n-k+2}(y).$$

The procedure just used also serves to shorten the computation in deriving the first addition formula for the "hyperbolic analogies."

§4. *The curve defined by the circular analogies.* Temple mentioned the fact that the equations

$$u_1=\cos x, \quad u_2=\sin x$$

represent the circle $C: u_1^2+u_2^2=1$. The first derived curve of C, namely

$$C_1: U_1=-\sin x, \quad U_2=\cos x$$

represents the same circle; the path-curve C_1 differing from the path curve C only in that points of C and C_1 with equal $x=$values are separated by a quadrant of the circle C.

Let C_p be the pth derived curve of C. Then from the following table one sees that the 4th derived curved of C is C:

C	C_1	C_2	C_3	C_4
$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$

§5. *The curve defined by the hyperbolic analogies.* Let Γ_p be the pth derived curve of the curve

$$\Gamma: u_i=u_i(x), \quad (i=1,\cdots,n),$$

where the u_i are Temple's solutions of equation (2). Then the following statement is evidently true: Γ_n is Γ and every Γ_p $(1 \leqq p < n)$ is transformable into Γ by a simple rigid motion.

§6. *Generalizations.* Since Temple's circular analogies and hyperbolic analogies include customary circular functions and hyperbolic functions, respectively, for the case $n=2$, his functions should properly be called *generalizations* rather than *analogies* of the customary circular and hyperbolic functions.

A further generalization may be made as follows. Consider the equation $(n \geqq 2)$

$$\frac{d^n y}{dx^n} + k^n y = 0, \quad (k = \text{const.} > 0).$$

Relative to it, define

$$v_p(x) = (-1)^{p-1} \cdot \frac{1}{n} \sum_{i=1}^{n} e_i^{p-1} e^{kl_i x}, \quad (p = 1, \cdots, n),$$

so that

$$v_p(x) = -\frac{1}{k} \frac{d}{dx} v_{p-1}(x), \quad (p = 2, \cdots, n).$$

One can readily show by Temple's methods that

$$v_1(x+y) = v_1(x)v_1(y) - (-1)^n \sum_{p=2}^{n} v_p(x)v_{n-p+2}(y),$$

so that Temple's addition formula is generalized.

Further, the Wronskian of the v's is a generalization of that of Temple's corresponding u's since

$$\begin{vmatrix} v_1 & v_2 & \cdots & v_n \\ v_1' & v_2' & \cdots & v_n' \\ \cdot & \cdot & & \cdot \\ v_1^{(n-1)} & v_2^{(n-1)} & & v_n^{(n-1)} \end{vmatrix} = k^{\frac{n(n-1)}{2}},$$

which reduces to 1 when $k=1$.

The corresponding generalizations relative to the equation $(n \geqq 2)$

$$\frac{d^n y}{dx^n} - k^n y = 0$$

can now be readily made. We leave this task to the reader.

A Note on a Diophantine Equation

By A. A. AUCOIN
Louisiana State University

We prove in this note the following theorem: The Diophantine equation

$$\sum_{j=1}^{m} \frac{a_j}{\sum_{i=1}^{n} \frac{b_{ij}}{x_{ij}}} = \frac{d}{\sum_{k=1}^{p} \frac{c_k}{y_k}} \tag{1}$$

where all the letters represent integers* and x_{ij}, y_k are unknown, has solutions and every solution is proportional to one of the infinitude of solutions given by

$$\begin{aligned} x_{ij} &= \alpha_{ij} dC(\beta)B(\alpha), \\ y_k &= \beta_k D(\beta) \sum_{j=1}^{m} \frac{a_j A_j(\alpha) B(\alpha)}{B_j(\alpha)} \end{aligned} \tag{2}$$

where

$$A_j(\alpha) = \prod_{i=1}^{n} \alpha_{ij},$$

$$B_j(\alpha) = \sum_{i=1}^{n} \frac{b_{ij} A_j(\alpha)}{\alpha_{ij}},$$

$$B(\alpha) = \prod_{j=1}^{m} B_j(\alpha), \quad C(\beta) = \prod_{k=1}^{p} \beta_k$$

$$D(\beta) = \sum_{k=1}^{p} \frac{c_k C(\beta)}{\beta_k}$$

and α_{ij}, $\beta_k (\neq 0)$ are arbitrary integers.

Proof: We may write (1) in the equivalent form

$$D(y) \sum_{j=1}^{m} \frac{a_j A_j(x) B(x)}{B_j(x)} = dC(y)B(x). \tag{3}$$

*We naturally exclude values of the coefficients which will make any denominator vanish identically.

If we let $x_{ij}=\alpha_{ij}t$, $y_k=\beta_k s$, (3) becomes

$$s^{p-1}\,t^{m(n-1)+1}\;D(\beta)\sum_{j=1}^{m}\frac{a_jA_j(\alpha)B(\alpha)}{B_j(\alpha)}=s^p\,t^{m(n-1)}dC(\beta)B(\alpha)\ ,$$

which is satisfied identically in the α's and β's if

$$s=D(\beta)\sum_{j=1}^{m}\frac{a_jA_j(\alpha)B(\alpha)}{B_j(\alpha)}\ ,\quad t=dC(\beta)\;B(\alpha).$$

Hence a solution of (1) is given by (2).

We now show that the parameters α_{ij}, β_k may be selected so that (2) gives a solution proportional to any existing one. Suppose that $x_{ij}=\lambda_{ij}$, $y_k=\mu_k$ is any solution of (1). If we choose $\alpha_{ij}=\lambda_{ij}$, $\beta_k=\mu_k$ we have that $s=t$ and (2) becomes $x_{ij}=\lambda_{ij}t$, $y_k=\mu_k t$ which is proportional to the given solution provided $t\neq 0$. But if $t=0$ then

$$B(\lambda)=\prod_{j=1}^{m}\;B_j(\lambda)=0$$

and hence some $B_j(\lambda)=0$. But if some $B_j(\lambda)=0$ then

$$\sum_{i=1}^{n}\;\frac{b_{ij}}{\lambda_{ij}}=0$$

since $A_i(\lambda)\neq 0$. This is impossible from (1).

If we had $a_1=d=1$, $a_2=a_3=\ldots.=0$, (1) reduces to*

$$\sum_{i=1}^{n}\frac{b_{i1}}{x_{i1}}=\sum_{k=1}^{p}\frac{c_k}{y_k}$$

*See, Dickson, History of the Theory of Numbers, vol. 2, pp. 690-691.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from April issue.)

The third mathematical journal was the *Mathematical Diary*, founded and edited by Dr. Adrain, of Columbia College, New York. An exact copy of the title page follows:

THE MATHEMATICAL DIARY;
containing
New Researches and Improvements
in the
MATHEMATICS:
with
COLLECTIONS OF QUESTIONS;
Proposed and Resolved by
INGENIOUS CORRESPONDENTS
In Quarterly Numbers
Conducted by
R. ADRAIN, LL.D., F.A. P.S., F.A. A.S. &c.
and Professor of Mathematics and Natural Philosophy
in Columbia College, New York

No. I

New York
Published by James Ryan
at the Classical and Mathematical Bookstore, 322 Broadway
J. Seymour, Printer

1825

The first number of this journal appeared in January, 1825. The Editor states in the preface that the object of the work is to excite the genius and industry of those who have a taste for mathematical studies, by affording them an opportunity of laying their speculations before the public in an advantageous manner; and thus to spread the knowledge of mathematics in a way that is both effectual and agreeable. He points out the fact that efforts put forth by a student to discover the solutions of new problems contributes most to the development of mathematical genius and that with this fact in view, many periodicals embracing mathematical investigations of one kind or another have been published in Great Britain; such, for example, are the *Ladies' Diary*, the *Gentlemen's Diary*, the *Mathematical Companion*, Dr. Hutton's *Miscellanea Curiosa*, *Leybourn's Mathematical Repository*, etc., whose influence on the state of mathematics in that country was very marked,—an influence acknowledged by men well informed on the subject, which advanced the knowledge of mathematics more rapidly than many works of greater magnitude. He notes that the English *Ladies' Diary* which was established in 1704 and published annually had among its editors Simpson and Hutton and among its contributors Emmerson, Simpson, Landen, Lawson, Vince, Hutton, Dalby, Major Henry Watson, Wales, and Mudge; and that Leybourn's *Repository* had among its supporters some of the best mathematicians in England, such, for example, as Barlow, Ivory, and Professor Wallace of Edinburgh. He also refers to the interest aroused by the *Diary's* predecessors published in the U. S.

The first contribution to the *Diary* is an article,—the editor calls it an essay,—"On the Rectification and Quadrature of the Circle," pp. 5-17.

After stating the definite requirements of the problems, and giving a brief history of the attempts to obtain an exact ratio of the circumference to the diameter, he shows the absurdity of some of the reasons assigned for our inability to obtain an exact ratio. In controverting the reason, that the curvature of a line involves some incompatibility between its measure and that of a straight line and that on this account the circumference of the circle is not susceptible of exact numerical comparison with the diameter; he takes as an illustration, the case of a cycloid, and calls attention to the fact that even straight lines themselves are not always capable of comparision in rational integers. The hypotenuse of the right-angled triangle affording a good illustration, he derives expressions for the sides of a right triangle in rational numbers and calls attention to an error in Emerson's Algebra, a work of great prominence in its day.

He also mentions some of the so called circle-squares among whom are Christian Logomontanus, the celebrated Danish Astronomer, who laid claim to having found π to be exactly equal to 3.14185 and Leistner, an officer of the French service who claimed that the exact value is 3844:1225. He then describes the methods of effecting the rectification of the circle by means of the Quadratrix and the Spiral of Archimedes, and gives a description of Clairaut's method of mechanically describing the latter. The article, which is quite modern in spirit, closes with an approximate calculation of π from a geometrical construction.

The second article is a critical review of Venturoli's *Elements of the Theory of Mechanics*.

The number concludes with 20 problems proposed for solution, the last of which is a prize problem, proposed by P. Fleming, C. E., surveyor of New York, and reads as follows:

> If from a point within a given circle, the three angles be observed which are subtended by the three contiguous arcs of the circumference, of which the two extreme arcs are given in magnitude, but not in position; it is required to determine the magnitude of the intermediate arc, and the situation of the point where the observations are taken.

The second number of the *Diary* is entitled "the Prize Number of Nathaniel Bowditch," so called because it contains Dr. Bowditch's prize solution of the "Prize Problem" proposed in No. I. The second number also contains the solutions of the problems in No. I., among which are two excellent solutions, one by Dr. Bowditch, the other by the Editor, of problem 19, proposed by the Editor. The problem reads as follows:

> It is required to investigate the nature of the curve described by a body projected obliquely along a given inclined plane, the resistance arising from friction being taken into consideration.

The number concludes with 18 problems proposed for solution, the last of which, proposed by Professor Adrain, reads as follows:

> It is required to investigate the path that ought to be described by a boat in crossing a river of given breadth from a given point on one side, to a given point on the other, so as to make the passage in the least time possible: supposing the simple velocity of the boat by the propelling power to be given; and that the velocity of the current being in the same direction with the parallel sides of the river, is variable and expressed by any given function of the perpendicular distance from the side of the river from which the boat sets out.

The third number contains an excellent solution of the above problem by Professor Theodore Strong, of Hamilton College, and 15 problems proposed for solution. The last of the fifteen problems was

proposed by a contributor under the pseudonym of "Eboracensis." It requires the determination of the perpendicular height of a hill, the angle of elevation of its vertex being measured from four different points in the plaine of its base, and the mutual distances between the four points being given. This problem was a "prize problem" and the prize for its solution was awarded to Henry J. Anderson of New York.

Number IV contains in addition to the solution of the above "prize problem", the solutions of the 15 problems proposed in No. I, and fifteen problems proposed for solution. The last is a "prize problem" proposed by Ferrand Benedict of Montazuma, New York. It required the triangle of minimum base and of given vertical angle such that its vertex lie on the circumference of a given circle, the position and one extremity of the base being given.

With number IV, Dr. Adrain's connections as Editor of the *Diary* ceased, the subsequent numbers being edited by the publisher, James Ryan. At this time, no information in the *Diary* to its readers either by the retiring Editor or his successor, Mr. James Ryan, the Publisher, of the change in Editorship is given.

However, on the 3rd page following the title page of Vol. II, Editor Ryan states "numbers XIII (the present number) of the *Mathematical Diary* completes the second volume. This periodical, begun in January, 1825, was conducted by Robert Adrain, LL.D., Professor of Mathematics and Natural Philosophy in Columbia College, New York, till 1826, when he was appointed Professor in Rutger's College, New-Brunswick, New Jersey; he was subsequently elected Professor in the University of Pennsylvania, Philadelphia, where he now resides.

"In consequence of Dr. Adrain's change of residence, I became editor and publisher of the *Diary*. Neither labour nor expense has been spared to make it useful and interesting to the public, and especially to those scientific gentlemen, who contributed to its support".

Mr. Ryan was an author of several text-books on mathematics, among them being an *Elementary Treatise* on *Algebra* and a *Differential* and *Integral Calculus*, the first Calculus written by an American author. He also wrote the *New American Grammar of Astronomy*.

Number V contains in addition to three solutions of the "prize problem" in No. IV, the solutions of the fourteen other problems proposed in No. IV and 18 problems proposed for solution, the last of which proposed by J. H. Swale, Liverpool, England is a "prize problem" and reads as follows:

> A right line is given in position; to assign the position of a point in the periphery of a circle (given in position and magnitude) such that thence drawing given and equal tangents, and demitting perpendiculars (upon the line given in position) from the extremities of the tangents; the rectangle of those perpendiculars shall be given, or a maximum or minimum.

Number VI is the Prize Number of Robert Adrain. It contains his solution of the prize problem, the solutions of the other 17 problems proposed in No. V and 21 problems proposed for solution. Dr. Adrain's prize solution was to be concluded in No. VII, but for some reason the last part of it was never published. In this number is also announced the death of a faithful contributor, viz., John Capp, of Harrisburg, Pennsylvania, whose death occurred February 14, 1826.

Of the 21 problems proposed in this number for solution in the next number, three are algebraic, requiring the solutions of systems of algebraic equations, five are in mechanics, one in geometrical progression, one in Diophantine Analysis, and the remainder are geometrical, involving the application of the Calculus. The prize problem, proposed by Robert Adrain, reads as follows:

> It is required to determine the time of the very small oscillations of an extremely slender, uniform, and inflexible rectilineal bar placed horizontally at rest on the surface of a uniform sphere; supposing the point of contact to be exceedingly near the middle of the bar, and the gravitation of the bar towards the sphere to be according to the law of nature.

Number VII is the Prize Number of Robert Adrain of Rutgers College, Henry J. Anderson of Columbia College, Nathaniel Bowditch of Harvard College, and Eugene Nulty of Philadelphia. Two solutions by Robert Adrain, three particular and three general solutions by Professor Anderson, two by Dr. Bowditch and one by Eugene Nulty, of the prize problem, are published. The prize was divided among the four contestants. Twenty-three new problems are published for solution; of these, four consist in solving systems of simultaneous algebraic equations, five are in mechanics, three are algebraic, and the remainder are geometrical. The twenty-third, a "prize problem", proposed by John Smith of Cincinnati, Ohio, is as follows:

> It is required to determine the greatest or least triangle, having its angular points in the circumference of three circles in the same plane not meeting each other, the three circles being given both in magnitude and position.

This number was printed by W. E. Dean. It appears, too, that the *Diary* had one lady contributor named Mary Bond. However, according to Dr. Daniel Kirkwood, of Indiana University, this was a pseudonym assumed by William Lenheart, in order that he might get more of his contributions published.

Number VIII was to be published July, 1827. It contains the solutions of the twenty three problems proposed in Number VII. In this number, appears for the first time the name of Benjamin Peirce, Jr., then a student at Harvard College, who afterwards became the

well known and honored professor of mathematics in that institution. He contributed a solution to the following problem:

Question XV (122). Suppose a small ball to roll down the quadrant of an ellipse by the attraction of gravitation, it is required to determine its position when its perpendicular velocity is a maximum.

He contributed other solutions, but this is the only one published.

Number VIII concludes with twenty-three problems proposed for solution, the last of which is a prize problem proposed by Dr. Anderson and reads as follows:

To determine the motion of a uniform heavy inflexible circular plate, placed originally in a position nearly vertical upon a horizontal plane, and then impelled in the direction of its plane; supposing the friction just sufficient to make the plate's circumference to roll without sliding, along its variable projection on the horizontal plane.

This number completes Volume I of the *Diary*.

Volume II begins with Number IX.

Number IX has as a frontispiece, a steel engraving of Joseph Louis Lagrange. This number in addition to the solutions of the problems proposed in Number VIII has twenty-two problems proposed for solution. The prize for the solution of the last prize problem was awarded to Eugene Nulty.

Number X contains the solutions of the problems proposed in Number IX. Twenty-eight pages are devoted to the discussion of the prize problem proposed in Number VIII. Twenty-four problems are proposed for solution in Number XI. In X, the Editor, Mr. Ryan, announces the publication of his *Differential and Integral Calculus*. On page 87, is a partial solution by Dr. Ingersoll Bowditch of Fermat's Last Theorem.

Number XI, which was announced in Number X to appear February 1830, contains the solutions of the twenty-one problems proposed in Number X. Some of the solutions are fine specimens of Diophantine Analysis contributed by William Lenheart at that time, apparently the most interested in that subject of all of the contributors. Benjamin Peirce contributed a number of fine solutions. Article XXVII of this number gives an account of the contents of the *Annales de Mathematiques pure et appliques*, by M. Gergonne, and the *Annals of Philosophy* by Richard Taylor and Richard Phillips. Also a notice of the publication of the Translation of Laplace, *Mechanique Celesti*, and other works.

This number concludes with twenty-three problems proposed for solution in Number XII.

Number XII contains the solutions of the problems proposed in Number XI and concludes with twenty-two problems proposed for

solution and fourteen pages devoted to *Evening Amusements*, or the Beauties of the Heavens displayed; in which several remarkable appearances to be observed on various nights in the Heavens, during the first six months of the year, 1831, are described.

Number XIII, has the following title page:

THE
MATHEMATICAL DIARY;
containing
COLLECTIONS OF ORIGINAL QUESTIONS
Proposed and Resolved by
INGENIOUS CORRESPONDENTS
To which are added
Important and Valuable Communications
Upon Various Branches
OF ABSTRACT SCIENCE,
A Memoir of Lagrange, with a Portrait,
Mathematical Dialogues, together with an
Index of American Mathematicians

Conducted by James Ryan, A.M.

New York
Published by James Ryan, A.M.
At his Mathematical Bookstore, 426 Broadway
G. F. Hopkins & Son, Printers
1832

In this number the editor, Mr. Ryan, states that the *Diary*, begun in January, 1825, was conducted by Robert Adrain LL. D., Professor of Mathematics and Natural Philosophy in Columbia College, New York, till 1826, when he was appointed Professor in Rutgers College, New Brunswick, New Jersey, and subsequently elected Professor in the University of Pennsylvania, where he resided at the time of publication of this number. The editors further observe that the contributors of the *Diary* enrolled the names of the most eminent American mathematicians of that time, viz., Dr. Bowditch, Dr. Adrain, Professor Nulty, Dr. Anderson, Professor Strong, and Professor O'Shannessy.

This last number of the *Diary* not only concluding the second volume, Vol. II, but also concluding the publication of the *Diary*, is the most important number of all, as it contains a little known article on

Perfect Numbers by Benjamin Peirce. This number contains in addition, the solutions of problems proposed in Number XII, Reply of Professor Strong to Article XXIV, pp. 263-265, Mr. Lenheart on Diophantine Analysis, Number 1, and Mr. Floy on the same subject Number 2, Professor Strong's solution of an unsolved problem, Mr. Colburn's solution of the unsolved Prize problem in Nash's Diary, Number III. Biographical sketch of J. L. Lagrange, by $\sum$ Horace, Decerptae by $\sum$, a dialogue in which Colburn, Zigma, (the writer), and a doctor, (Dr. Anderson), are the characters.

Review of Professor Anderson's paper by $\sum$, miscellaneous notes by $\sum$, twenty-six new questions proposed for solution, and an Index of all the contributors to the *Diary*. Mr. Samuel Ward, 3rd, of New York, proposed in this number and offered $100 for a correct solution of the following problem:

It is required to determine all the oscillations of a hollow hemiellipsoidal cup, and given quantity of homogenious fluid contained in it, when the cup rocks without sliding about the shorter axis upon a fixed horizontal plane.*

*The opening part of the October installment of the Finkel series will deal with the little known paper on Perfect Numbers by Benjamin Peirce.

'Some years ago this department appointed a group of young instructors. One of them left us at the end of the first year. He was complained of more than almost any other man in the whole institution. His students weren't 'getting anything from him' and at regular intervals threatened to leave his classes; the dean was up in arms against him; even those who liked him reported him to be a poor teacher. As one of the senior members of the department, I decided to get some first-hand information; accordingly I visited three of his classes. The first two rows of seats were unoccupied. He lectured. It seemed that he centered his attention on the empty rows. He was barely audible. I concluded that I had there and then witnessed the world's poorest teaching. Needless to say, at the end of the year, he was asked to resign. However, I haven't reached the moral of the story yet. Our final Examinations are departmental. The papers of his students were collected and graded by two veteran teachers. And here is the surprise: As a class, his students ranked highest—first on the list of 12 sections. Well, sir, what is good teaching"?

—From a narrative quoted by Dr. Joseph Seidlin in his "*A Critical Study of the Teaching of Elementary College Mathematics*". Published by Bureau of Publications, Teachers College, Columbia University.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

We Discover the Meaning of Curvature

By EDWIN G. OLDS
Carnegie Institute of Technology

It is a matter of considerable satisfaction to any teacher to feel that he has presented a new topic to his class in the best expository style, with the steps of his demonstration properly arranged and supported. He leaves his class room with the conclusion that he has covered a lot of ground in a very neat fashion. Much of his satisfaction evaporates or changes to irritation when, at the next meeting of the class, he finds that his fine presentation has failed to produce any great increase in knowledge on the part of his students. Then, if he is an experienced teacher, he realizes that he has accumulated another bit of experimental evidence of the weakness of the lecture method as a tool for promoting learning.

It is not the purpose of this note to provide any detailed appraisal of the lecture method. It has virtues as well as vices. Usually, however, it seems to have the characteristic indicated above; it gives the teacher the illusion of making rapid progress. Another method, which produces the opposite illusion, is the so-called "heuristic" method. In a class where this method is in use teacher and students discuss together the topic of the day, with the students expressing their thoughts as ideas develop. The teacher directs the discussion and sometimes accelerates it by judicious suggestions, but by no means dominates it. Often very little ground is covered, but, unlike the situation with the lecture method, the advanced position is thoroughly consolidated. The purpose of this note is to outline the manner in which the heuristic method has been used to develop the concept of Curvature with a class in Calculus.

"Most of you fellows drive automobiles. On the roads around here there are plenty of curves. Some are marked "Curve" and others are marked "Sharp Curve"; very little of the road is absolutely straight. What's the difference between a sharp curve and a gentle one?"

Kenworth suggests that you turn more on a sharp curve.

"Do you mean that you make a greater change in direction?"

After some consideration it is decided that it is possible to change direction from East to North by a long, gentle curve or by turning an abrupt corner. Some one suggests that the difference in a curve lies in the distance one travels in making the turn.

"Then how do you think curvature should be defined?"

Stanton thinks it should be defined as the ratio of the change in distance to the change in angle, but, after a suggestion that the curvature of a line should be zero, it is decided that the above ratio should be inverted.

The teacher draws a curve on the blackboard and locates a point P. Then he inquires, "How would you define the curvature at P?"

One of the boys asks to have a point, Q, located and then says that the curvature of the segment PQ should be defined as the ratio of the change in direction to the length of PQ.

"But the curvature might not be the same at P. Of course we may call the ratio of change in direction to length of arc, the *average curvature*, but what about the curvature *at* P?"

Johnson suggests that we might take the limit as Q approaches P. This meets the approval of the class. Then, with prompting by various students the figure is properly labeled, the average value of K is written as $\Delta\Phi/\Delta s$ and the expression, $d\Phi/ds$ is obtained for the curvature, K, at point P.

"Is there any curve for which the curvature is constant?"

The circle is given as an example and its curvature is found to be the reciprocal of the radius. The teacher then tells the class that, for this reason, the reciprocal of *curvature* is called *radius of curvature*.

"Suppose we have the parabola, $y^2=4x$; how would we find the curvature at $x=1$?"

Some of the students wanted to approximate the curve by means of a circle but the teacher decided that this should be deferred. The idea of expressing the formula for curvature in rectangular coordinates had to be produced by the teacher but, after the attention was focused on this problem, the derivation moved quite smoothly. Then the formula was applied to the parabola and a numerical result obtained.

It seems unnecessary to outline in detail the remainder of the lesson. The teacher and pupils returned to the question of approximating the parabola by a circle; the osculating circle was briefly alluded to and one was drawn for the given parabola and designated point. In passing, it might be reported that the teacher was secretly amused because no one knew the meaning of the word, "osculate." The lesson

closed with a suggestion that a knowledge of the meaning of this word was a necessary part of any young man's social education.

The author makes no claim that this is the best way to introduce the topic of Curvature; perhaps many of the readers will have the opinion that it is not even a good way. They may agree, however, that it is a better way than the procedure where the teacher presents a nicely logical and highly polished lecture in text book style and the students listen or dream, but do little thinking.

Editor's Note. It is not within the province of the editors' duties to comment upon contributors' methods of developing or conducting class presentations. We do not, however, step over the bounds of that province in exhorting the readers of Professor Old's article to "go and do likewise". We should like to see the *National Mathematics Magazine* used as "proving grounds" for the building up of a source book in different methods of and approaches to the teaching of college mathematics. However much of importance we may attach to the underlying philosophies or principles of the teaching of mathematics, we must not overlook the potential aid to teaching in articles such as this one by Professor Olds.

The Drawing of Quadric Surfaces

By Neil Little
Purdue University

The majority of the drawings of quadric surfaces shown in texts on analytic geometry and calculus are revealed upon analysis to contain a number of inaccuracies. This fact is either not known to mathematicians in general, or, if known, is regarded to be of no consequence. On the assumption that mathematicians do not knowingly present inaccurate drawings in their texts, this paper proposes to do two things; first, to point out some of those inaccuracies, and second, to show how accurate drawings can be made.

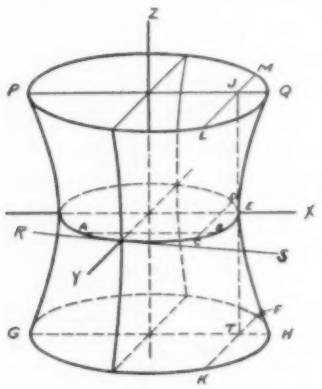


Fig 1

In Fig. 1 is shown a drawing of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

as commonly shown in texts. The trace of this surface in the xy-plane is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ,\ z=0.$$

This curve is symmetric to the $x-$ and $y-$axes. It seems reasonable to suppose that the curve as shown in the drawing should appear to

be symmetric to these axes as drawn, that is, chords parallel to the $x-$axis should be bisected by the $y-$axis, and those parallel to the $y-$axis should be bisected by the $x-$axis. Observe, however, lines AB and CD in Fig. 1. It is quite apparent that the requirement mentioned is not satisfied. It is even possible on such a drawing to draw chords parallel to the $y-$axis which do not intersect the $x-$axis at all, and chords parallel to the $x-$ axis which do not intersect the $y-$ axis.

Again, tangents drawn to the trace in the $xy-$plane at the points where the curve crosses either axis should be parallel to the other axis. But in Figure 1, line RS is obviously not parallel to the $x-$axis, and line JT, instead of being parallel to the $y-$axis is actually parallel to the $z-$axis.

Such discrepancies exist in all of the curves drawn as the traces of the surface in planes parallel to the $xy-$ or $yz-$planes. They are particularly noticeable if one tries to make all the parts of the drawing consistent with one another. For example, in the surface just given, the trace in the plane $x=a$ is a pair of straight lines. These should be capable of being found on the drawing as follows. At point E where the surface crosses the positive $x-$axis, draw a line parallel to the $z-$axis intersecting PQ and GH (parallels to the $x-$axis) at J and T respectively. At J and T draw lines parallel to the $y-$axis intersecting ellipses as shown at points L and M, and F and K, respectively. Now if the figure were correctly drawn, points L, E, and F would be collinear, as would also points M, E, and K. It is apparent even to the unaided eye that this requirement is not satisfied.

Precisely this trouble is encountered with inaccurately constructed drawings if one attempts to draw two surfaces using the same set of axes for both and to show their curve of intersection. A classic example of this type of error exists in one of our oldest and most widely used texts on the calculus, wherein the curve of intersection of the surfaces $x^2+y^2=az$ and $x^2+y^2=2ax$ is shown as a skew curve, whereas the simplest of calculations shows it to be an ellipse.

Again, in most drawings, the trace in the $xz-$plane is shown as if it were the curve of visibility (also called bounding curve, outline curve, and apparent contour) of the surface. Here one must realize that the drawing of any surface is the drawing of the projections on a plane of strategic curves of the surface*. If a projecting cylinder is used, the result is an oblique or orthographic drawing, if a projecting cone is used, the result is a perspective drawing. In order for the curve

*For a more detailed study of this subject see Neil Little, "An Analytical Study of the Non-Perspective Picturization of Quadric Surfaces", The American Mathematical Monthly, Vol. 44, 1937, pp. 292-305, and W. H. Roever, "Analytical Treatment of Perspection with Bearing on Picturization", Ibid, Vol. 45, 1938, pp. 278-293.

of visibility of a surface to be the trace in the $xz-$plane, it would be necessary to use an orthographic projection upon a plane parallel to the $xz-$plane. But if such a projection were used, the $y-$axis would appear as a point instead of as a line, and traces in the $xy-$ and $yz-$ planes and in all planes parallel thereto would appear as straight lines or segments of straight lines. In Figure 2 is shown how the surface of Figure 1 would appear under such circumstances.

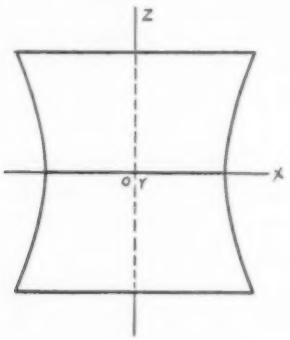


Fig. 2.

There are other errors which occur less frequently than those already mentioned. A complete discussion of such would take up an unwarranted amount of space, but there are two which perhaps should be mentioned. One is the showing of such curves as the traces of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

in the planes $z=k$ and $z=-k$ as ellipses of different eccentricities. This is without justification except in a drawing in full perspective. Another error is to draw a surface in such a way that a curve of visibility is obviously needed but is not drawn in.

The following is an account of a mechanical method of making correct drawings. I have used this mehtod in my own classes for a number of years and found it quite satisfactory. It can be applied to the making of either oblique or orthographic drawings. I usually ask my students to make isometric drawings. Suppose the surface to be drawn has the equation

$$\frac{x^2}{25}+\frac{y^2}{9}-\frac{z^2}{16}=1.$$

For an isometric drawing the axes are drawn making angles of 120 degrees with each other. The equation of the trace in the $xy-$plane is

$$\frac{x^2}{25}+\frac{y^2}{9}=1.$$

Assigning values to x, say, and computing the corresponding values of y we get the coordinates of a number of points. These points are then plotted, distances equal to the values of x being measured from the $y-$axis *along parallels to the* $x-$*axis*, and distances equal to the values of y being measured from the $x-$axis *along parallels to the* $y-$*axis* Connecting these points by a smooth curve we have the lower curve shown in Figure 3. For the trace in the plane $z=6$, say, we get values for x and y from the equation

$$\frac{x^2}{25}+\frac{y^2}{9}=\frac{13}{4}.$$

To plot these points we measure 6 units up on the $z-$axis, draw lines parallel to the $x-$ and $y-$axes and use those lines for the coordinate axes. The result is the upper curve shown in Figure 3. Proceeding in

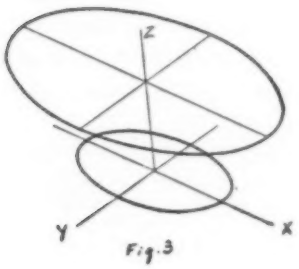


Fig. 3

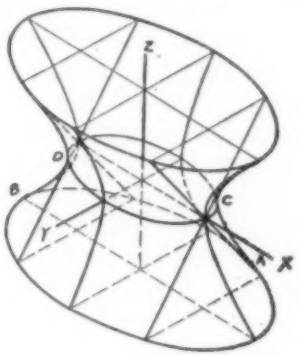


Fig. 4

similar fashion we draw the traces in the planes $x=0$, $y=0$, $z=-6$, and $x=\pm 5$. The completed figure appears as in Figure 4, with the curve of visibility drawn in and hidden lines dotted. The reader is invited to apply any test of accuracy to this drawing.

The curve of visibility will probably have to be "guessed in" by the freshman. In doing this, a cardinal principle to be followed is that the curve will be tangent to every curve on the surface which would be partly visible and partly hidden were the entire surface drawn.

Thus in Figure 4, the curve of visibility is tangent to the trace in the plane $z=-6$ at points A and B, to the trace in the plane $z=0$ at C and D, etc. Notice that it is at these points of tangency that a curve on the surface changes from a visible one to an invisible one. The curve of visibility is tangent to the trace in the plane $z=6$ because that trace would be partially hidden if the entire surface were con-

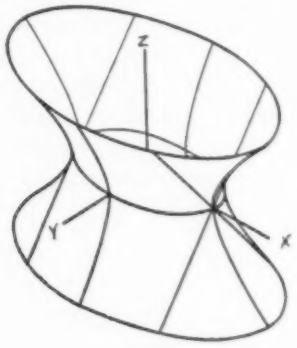


Fig 5

sidered. It is not tangent to the trace in the $yz-$plane because one branch of that curve is wholly visible, the other wholly hidden. The drawing looks better if hidden lines are not shown, as in Figure 5.

The curve of intersection of two surfaces is easily handled. The equations of two of the projecting cylinders of the intersection are determined in the ordinary way. If the two equations both contain x, say, then values are given to x, the corresponding values of y and z determined, and the points plotted.

This method of drawing is greatly expedited by the use of isometric drawing paper which can be purchased from the manufacturers of this type of paper.

Mathematical World News

Edited by
L. J. ADAMS

The Engineering and Mathematics Section of the Southern California Junior College Conference met at the Los Angeles City College on April 13, 1940. Two addresses were delivered:

1. *How Well Are the Young Men of Today Prepared for Places in Modern Industry?* Mr. C. T. Reid, Director of Education, Douglas Aircraft Co.
2. *Existence Theorems.* Dr. W. M. Whyburn, University of California at Los Angeles.

The section decided to divide into two groups, and elected Mr. L. J. Adams (Santa Monica Junior College) chairman of the new mathematics section, and Mr. C. Garrett (Pasadena Junior College) chairman of the new engineering section.

The American Association for the Advancement of Science has as a part of its annual exhibition a display of books known as The Science Library which is limited to books published during the year. These books are grouped and classified according to the sections of the Association. They include mathematics textbooks for high schools, colleges and universities.

The Indiana Section of the Mathematical Association of America was scheduled to meet at Earlham College in Richmond, Indiana on May 3-4, 1940. Addresses scheduled included:

1. *Least Time and Least Action.* Dr. Karl Darrow.
2. *Fibonacci Sequences and Certain Irrational Numbers.* Professor L. S. Shively.

The papers to be presented were:

1. *Condition for Conformality in Non-Euclidean Geometry.* Professor H. Randolph Pyle, Earlham College.
2. *The Drawing of Quadratic Surfaces.* Mr. Neil Little, Purdue University.
3. *The Transits of Mecury.* Professor K. P. Williams, Indiana University.

4. *Diagnostic Mathematics. Testing in Purdue University.* Dr. M. W. Keller and Dr. D. R. Shreve, Purdue University.
5. *The Real Shape of Plane Cubics.* Professor G. H. Graves, Purdue University.
6. *Methods of Teaching Irrational Numbers.* Dr. R. J. Duffin, Purdue University.
7. *Generators for Commutative Algebras.* Dr. G. W. Whaples, Indiana University.

The officers of the Northern California section of the M. A. A. for 1940-41 are: Chairman, Professor Sophia H. Levy (University of California); Vice-chairman, Professor F. R. Morris (Fresno State College); Secretary-Treasurer, Professor H. M. Bacon (Stanford University); Associate Editor for the California Journal of Secondary Education, Mrs. Ruth G. Sumner (Oakland High School).

The officers of the Southern California section of the M. A. A. for 1940-41 are: Chairman, Professor O. W. Albert (University of Redlands); Vice-chairman, Mr L. J. Adams (Santa Monica Junior College); Secretary, Professor P. H. Daus (University of California at Los Angeles).

The 1940 Catalog of the firm of Bernard Quaritch of London, England, includes announcements of a book, *Science Since 1500*, by H. T. Pledge. This is a short history of mathematics, physics, chemistry and biology since 1500.

The Hirschwaldsche Buchhandlung of Berlin NW7, Unter den Linden 60, announces a 25% reduction in price on German books and magazines delivered to non-German addresses.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

Late Solutions: Nos. 325, 326, 329, 331, 336, 337 by *Johannes Mahrenholz*; Nos. 326, 331 by *R. Goormaghtigh*; No. 325 by *W. L. Roberts*.

No. 314. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given points A and B outside a given circle. Determine a point X on the circle such that AX and BX will have equal chords within the circle.

Solution by the *Proposer*.

Since the chord segments will be equal, if AX and BX make equal angles with the tangent to circle O at X, this is recognized as the problem of Alhazen.

Using rectangular coordinates with OB as the $\alpha-$axis and setting $A=(a,b)$; $B=(c,O)$; $X=(\alpha,\beta)$; we have the equations of the circle O: $\alpha^2+\beta^2=r^2$; OX: $y=\beta x/\alpha$; AX: $(y-\beta)(a-\alpha)=(b-\beta)(x-\alpha)$; BX: $(y-\beta)(c-\alpha)=-\beta(x-\alpha)$.

Further:

$$[(b-\beta)/(a-\alpha)-\beta/\alpha]/[1+\beta(b-\beta)/\alpha(a-\alpha)]=\tan(AXO)=\tan(BXO)$$
$$=[\beta/\alpha+\beta(c-\alpha)]/[1-\beta^2/\alpha(c-\alpha)].$$

Thus

$$(\beta^2-\alpha^2)[bc-b\alpha-(a+c)\beta+2\alpha\beta]+2\alpha\beta[ac-(a+c)\alpha+b\beta-(\beta^2-\alpha^2)]=0,$$

or

$$bc(\beta^2-\alpha^2)+2ac\alpha\beta-r^2(a+c)\beta+br^2\alpha=0. \tag{1}$$

This is evidently an equilateral hyperbola which passes through O. Using the equation of the circle with (1), we obtain two solutions when A and B are external to the circle and four solutions when A and B are within the circle.

According to Robert Simson, who gave a geometric proof in his Sectionum Conicarum, 1750, p. 223, the problem was solved by the Arabian geometer Alhazen (c. 980-1038) in his Optics, Book 5, prop. 39. For a bibliography of the problem see Simson above; F. G. M. Exercises de géométrie, 5th ed. p. 720; L'Intermédiare des mathématiciens, 1900, p. 263.

Editors Note: The character of the problem indicates an effective solution by means of the complex variable. See an article by *G. A. Bingley*, American Mathematical Monthly, 1926, pp 418-421. The following construction is given there.

Let A and B be the points inverse to A and B with respect to the given circle, and let M be their midpoint. Through M draw two perpendicular lines, one of which is parallel to the bisector of angle AOB. With these lines as asymptotes construct the hyperbola passing through O. The real intersections of the curve and circle are the required points X.

No. 328. Proposed by *J. Rosenbaum*, Bloomfield, Conn.

Denoting the rth coefficient in the expansion $(a+b)^n$ by (n,r) and assuming that n is a positive integer and that $n>r+k-2$, show that

$$(n,r)=(n+k,\ r+k)-(k,\ 2)(n+k-1,\ r+k)+(k,\ 3)(n+k-2,\ r+k)$$
$$-(k,\ 4)(n+k-3,\ r+k)+\cdots+(-1)^k\ (n,\ r+k).$$

Solution by *Johannes Mahrenholtz*, Cottbus, Germany.

We prefer the customary notation

$$\left(\begin{array}{c} n \\ r-1 \end{array} \right)$$

instead of (n, r).

If in the identity

$$(1-x)^{-r-k-1}\ (1-x)^k=(1-x)^{-r-1}$$

there be substituted the well-known formulae

$$(1-x)^{-n-1}=\sum_{j=0}^{\infty}\left(\begin{array}{c} n+j \\ n \end{array} \right)x^j,\quad (1-x)^n=\sum_{j=0}^{n}(-1)^j\left(\begin{array}{c} n \\ j \end{array} \right)x^j$$

and the coefficients of x^{n-r} be then equated, there results the relation

$$\binom{n+k}{r+k}-\binom{k}{1}\binom{n+k-1}{r+k}+\binom{k}{2}\binom{n+k-2}{r+k}-\binom{k}{3}$$

$$\binom{n+k+3}{r+k}+\cdots+(-1)^k\binom{n}{r+k}=\binom{n}{n-r}.$$

Since $$\binom{n}{n-r}=\binom{n}{r},$$

this is the proposed equation with $r-1$ replaced by r.

Note by the Editors. The solution usually given is independent of infinite series and involves the use of

$$\binom{n}{r-1}=\binom{n+1}{r}-\binom{n}{r},$$

which is easily verified. Let each term on the right be again expressed as a difference of binomial coefficients. We obtain

$$\binom{n}{r-1}=\binom{n+2}{r+1}-2\binom{n+1}{r+1}+\binom{n}{r+1}.$$

If this process be repeated $k-2$ more times, we shall have the proposed formula

$$\binom{n}{r-1}=\binom{n+k}{r+k+1}-\binom{k}{1}\binom{n+k-1}{r+k+1}$$

$$+\binom{k}{2}\binom{n+k-2}{r+k-1}-\cdots\pm\binom{n}{r+k-1}.$$

No. 332. Proposed by *C. W. Trigg*, Los Angeles City College.

On side AB of the triangle ABC as base are constructed two isosceles triangles with 120° vertex angles, C' exterior to ABC and C'' interior to ABC. The same procedure is followed on the other two sides of ABC.

(1) $A'B'C'$ and $A''B''C''$ are equilateral.
(2) ABC, $A'B'C'$, and $A''B''C''$ have the same centroid.

(3) Find the sides of the equilateral triangles in terms of the sides of the given triangle.

(4) Under what circumstances will the vertices of one of the equilateral triangles fall upon the sides of the other?

Solution by *D. L. Mackay*, Evander Childs High School, New York.

(1) A',B',C', are the centers of the exterior equilateral triangles BA_1C, CB_1A, and AC_1B. Hence $B'C/AC = 1/\sqrt{3} = A'C/BC$. Since angle $B'CA' = C+60° =$ angle ACA_1, triangles ACA_1 and $B'CA'$ are similar. In like manner, triangles ACC_1 and $B'AC'$, BCC_1 and $C'BA'$ are similar. Then $B'A'/AA_1 = B'C'/CC_1 = C'A'/CC_1 = 1\sqrt{3}$.

Now triangles B_1CB and ACA_1 are congruent, for angle $B_1CB = C+60° =$ angle ACA_1 and the including sides are equal. Hence $BB_1 = AA_1$ and, in like manner, $BB_1 = CC_1$. Hence $B'A' = B'C' = C'A'$. Similarly it can be shown that $A''B''C''$ is equilateral.

(2) Obviously, $AC''BC''$ is a parallelogram. Triangles $C''AB'$ and ABC are similar, since angles BAC and $C''AB'$ are equal. Thus $AB/AC = AC''/AB'$. It follows that the angle formed by the homologous sides $C''B'$ and BC is equal to angle $C''AB$ and hence to angle $A'CB$. Thus $A'C$ is parallel to $C''B'$ and likewise $C''A'$ is parallel to CB'. Hence $C''B'CA'$ is a parallelogram and N, the intersection of $B'A'$ and $C''C$, is the midpoint of each.

In triangle $CC''C'$, let the medians CM and $C'N$ intersect at G, a trisection point of each. Now CM is a median of triangle ABC and $C'N$ is a median of triangle $A'B'C'$. Hence the two triangles have the same centroid. The same method indicates a common centroid for ABC and $A''B''C''$.

(3)
$$\begin{aligned}(A'B')^2 &= (B'C)^2+(A'C)^2-2(B'C)(A'C)\cos(C+60°)\\ &= (b^2+a^2)/3-(ab/3)(\cos C-\sin C\sqrt{3})\\ &= (\tfrac{1}{6})(a^2+b^2+c^2+4\sqrt{3}s(s-a)(s-b)(s-c).\end{aligned}$$

The expression for $(A''B'')^2$ is the same except for a change in sign of the radical.

(4) If B'' is on $B'C'$, then $B'C'$ will be the perpendicular bisector of AC and $CC' = AC' = BC'$. Then $C'A'$ is the perpendicular bisector of BC and A'' will lie on $A'C'$. Then angle $C'CB = B+30°$, angle $ACC' = A+30°$ and thus $C = 120°$ with B', C, and A' collinear.

If one seeks the condition that C'' lie on $A'B'$, then, using rectangular coordinates with $A = (O,O)$; $B = (c,O)$, we have:

$$\begin{vmatrix} c-(a/\sqrt{3})\cos(B+30° & (a/\sqrt{3})\sin(B+30°) & 1\\ (b/\sqrt{3})\cos(A+30°) & (b/\sqrt{3})\sin(A+30°) & 1\\ c/2 & c/2\sqrt{3} & 1\end{vmatrix} = 0,$$

or $(bc \sin A + ac \sin B) + c\sqrt{3}(b \cos A + a \cos B) - ab \sin(A+B)$
$$-ab\sqrt{3} \cos(A+B) - c^2\sqrt{3} = 0.$$

Since $bc \sin A = ac \sin B = ab \sin C$, $c = b \cos A + a \cos B$, $\cos(A+B) = -\cos C$, $\sin C + \sqrt{3} \cos C = 0$, then $C = 120°$. Accordingly, if any angle of triangle ABC equals 120°, triangle $A''B''C''$ is inscribed to triangle $A'B'C'$.

Also solved by *Johannes Mahrenholz* and the Proposer. A reference is given by Mahrenholz to Fischer, ein geometrischer Satz, Archiv der Mathematik und Physik, Series I, Volume 40, 1863, p. 460.

No. 334. Proposed by *H. L. Smith*, Louisiana State University.

Consider the set of all points (x,y) which satisfy the inequalities

$$x^2 + y^2 \leqq r^2, \; r > 0$$
$$|x+y| + |x-y| \leqq 2.$$

Let $A(r)$ denote the area of that set. Determine the minimum value of $r/A(r)$.

Solution by *C. W. Trigg*, Los Angeles City College.

The first inequality defines the set of points on or within the circle of radius r. The second defines* the set of points on or within the square, center at origin, sides of length 2, parallel to the axes. If $r \geqq \sqrt{2}$, the set of points common to the two figures is the set of points in the square; whence $r/A(r) = r/4$. This ratio is $\sqrt{2}/4$ for $r = \sqrt{2}$, and increases with r. If $r \leqq 1$, the set (x,y) is the set of points in the circle; whence $r/A(r) = r/\Pi r^2 = 1/\Pi r$. This ratio is $1/\Pi$ for $r = 1$ and increases as r decreases.

Hence we seek a minimum value for $F = r/A(r)$ within the range $1 < r < \sqrt{2}$. For an arbitrary circle within this range, $A(r)$ comprises the area of the circle diminished by four segments. Indicate by 2θ the central angle subtending one of the segments. Then we have

$$A(r) = \Pi r^2 - 4r^2\theta + 4 \tan\theta, \; r = \sec\theta, \; 0 < \theta < \Pi/4;$$
$$1/F = A(r)/r = (\Pi - 4\theta) \sec\theta + 4 \sin\theta;$$
$$dF/d\theta = -F^2[(\Pi - 4\theta) \sec\theta \tan\theta - 4 \sec\theta + 4 \cos\theta].$$

This is easily reduced to the relation

$$dF/d\theta = -F^2 \sec\theta \tan\theta(\Pi - 4\theta - 2 \sin 2\theta),$$

*Readers who may be interested in the interpretation in analytic geometry of equations and inequalities involving absolute values are referred to C. O. Oakley, *Equations of Polygons*, Amer. Math. Monthly, v. 42, Oct. 1935, pp. 476-487.—ED.

which will vanish for $0<\theta<\Pi/4$ if and only if $\Pi-4\theta=2\sin 2\theta$, or

$$\varphi=\cos\varphi, \text{ where } \varphi=\Pi/2-2\theta.$$

By the usual methods this yields the single root $\varphi=.73909$ radians. Thus $\theta=.41585$ radians or 23°49′36″, $r=\sec\theta=1.0932$, and F has the minimum value $1.0932/3.5337=.3094$.

No. 338. Proposed by *Daniel Arany*, Budapest, Hungary.

Put $a=\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix}$ and $b=\begin{vmatrix} b_{11}b_{12}b_{13} \\ b_{21}b_{22}b_{23} \\ b_{31}b_{32}b_{33} \end{vmatrix}$;

further

$$d=\begin{vmatrix} a_{12}b_{13}-a_{13}b_{12} & a_{22}b_{23}-a_{23}b_{22} & a_{32}b_{33}-a_{33}b_{32} \\ a_{13}b_{11}-a_{11}b_{13} & a_{23}b_{21}-a_{21}b_{23} & a_{33}b_{31}-a_{31}b_{33} \\ a_{11}b_{12}-a_{12}b_{11} & a_{21}b_{22}-a_{22}b_{21} & a_{31}b_{32}-a_{32}b_{31} \end{vmatrix}$$

and

$$D=\begin{vmatrix} A_{12}B_{13}-A_{13}B_{12} & A_{22}B_{23}-A_{23}B_{22} & A_{32}B_{33}-A_{33}B_{32} \\ A_{13}B_{11}-A_{11}B_{13} & A_{23}B_{21}-A_{21}B_{23} & A_{33}B_{31}-A_{31}B_{33} \\ A_{11}B_{12}-A_{12}B_{11} & A_{21}B_{22}-A_{22}B_{21} & A_{31}B_{32}-A_{32}B_{31} \end{vmatrix},$$

where A_{ik} and B_{ik} are the cofactors of a_{ik} and b_{ik} respectively. If $a\neq 0$, $b\neq 0$, show that $D=abd$.

Solution by the *Editors*.

Multiply b and d, using the well known theorem on multiplication of determinants, and substitute the values of B_{ik} as they occur. The result is: $bd=$

$$\begin{vmatrix} 0 & -a_{21}B_{31}-a_{22}B_{32}-a_{23}B_{33} & C_1 \\ a_{11}B_{31}+a_{12}B_{32}+a_{13}B_{33} & 0 & C_2 \\ -a_{11}B_{21}-a_{12}B_{22}-a_{13}B_{23} & a_{21}B_{11}+a_{22}B_{12}+a_{23}B_{13} & 0 \end{vmatrix}$$

in which $$C_1=a_{31}B_{21}+a_{32}B_{22}+a_{33}B_{23}$$

and $$C_2=-a_{31}B_{11}-a_{32}B_{12}-a_{33}B_{13}$$

Let A be the determinant obtained from a by replacing a_{ik} by A_{ik}. Noting that the effect of interchanging the A's and B's in each element of D is to change the sign of D, one may write immediately the product of A and $-D$ by analogy with the above. It is: $-AD=$

$$\begin{vmatrix} 0 & -B_{21}\alpha_{31}-B_{22}\alpha_{32}-B_{23}\alpha_{33} & D_1 \\ B_{11}\alpha_{31}+B_{12}\alpha_{32}+B_{13}\alpha_{33} & 0 & D_2 \\ -B_{11}\alpha_{21}-B_{12}\alpha_{22}-B_{13}\alpha_{23} & B_{21}\alpha_{11}+B_{22}\alpha_{12}+B_{23}\alpha_{13} & 0 \end{vmatrix}$$

in which $$D_1 = B_{31}\alpha_{21} + B_{32}\alpha_{22} + B_{33}\alpha_{23}$$

and $$D_2 = -B_{31}\alpha_{11} - B_{32}\alpha_{12} - B_{33}\alpha_{13}$$

where α_{ik} is the cofactor of A_{ik} in A. It is easy to show that $A = a^2$ and that $\alpha_{ik} = a \cdot a_{ik}$. Thus one finds $-A \cdot D = -a^2 \cdot D =$

$$a^3 \begin{vmatrix} 0 & -a_{31}B_{21} - a_{32}B_{22} - a_{33}B_2 & E_1 \\ a_{31}B_{11} + a_{32}B_{12} + a_{33}B_{13} & 0 & E_2 \\ -a_{21}B_{11} - a_{22}B_{12} - a_{23}B_{13} & a_{11}B_{21} + a_{12}B_{22} + a_{13}B_{23} & 0 \end{vmatrix}$$

in which $$E_1 = a_{21}B_{31} + a_{22}B_{32} + a_{23}B_{33}$$

and $$E_2 = -a_{11}B_{31} - a_{12}B_{32} - a_{13}B_{33}$$

Noting the correspondences of elements, one sees easily that the determinants given in the expressions for bd and $-a^2D$ are numerically equal but of opposite signs. Therefore $-a^2D = a^3(-bd)$ or $D = abd$.

No. 339. Proposed by *Virgil S. Mallory*, New Jersey State Teachers College at Montclair.

Jack Donovan was killed on a lonely road two miles from Trenton, New Jersey, at 3:30 A.M., March 17, 1933. Shorty Malone, Tony Verelli, Hank Rogers, Joey Freiberg and Red Johnson were arrested a week later in Philadelphia, Penn. Under cross examination, each of these men made four simple statements, three of which are true, and one false. One of these men killed Donovan: deduce which.

Shorty said: I was in Chicago when Donovan was murdered. I never killed anyone. Red is the guilty man. Joey and I are pals.

Hank said: I did not kill Donovan. I never owned a revolver in my life. Red knows me. I was in Philadelphia March 17th.

Tony said: Hank lied when he said he never owned a revolver. The murder was committed on St. Patrick's day. Shorty was in Chicago at that time. One of us five is guilty.

Joey said: I did not kill Donovan. Red has never been in Trenton. I never saw Shorty before. Hank was with me in Philadelphia March 17th.

Red said: I did not kill Donovan. I have never been in Trenton. I do not know Hank. Shorty lied when he said I'm guilty.

Solution by *W. L. Roberts*, Student, Colgate University.

Since it is known that Red issues but one falsehood, it follows that his first and fourth statements, which cover the same fact, are both true. Thus the testimony by Shorty that Red is the guilty man is seen to be false, thereby leaving his other three statements as

true. The first one of these is confirmed by Tony's third statement, and when Tony repeats two details of the hypothesis, three verities have been enumerated. Tony's first statement must then be false, whereby Hank's claim of never having owned a revolver is true.

Turning now to Shorty's last, and true, statement that he and Joey are pals, the falsity of the third statement made by Joey is pointed out, whereupon Joey's other three statements are acknowledged as true. Thus the testimony by Red that he has never been in Trenton is upheld, which leaves his remaining statement a falsehood. We now see that Hank tells the truth when he says that Red knows him. It is also true that Hank was in Philadelphia the 17th of March being supported by a true statement by Joey. This indicates that Hank's first assertion is a lie and he is thereby branded as the murderer.

Also solved by *Eli Borok, George W. Campbell, Richard W. Davies, E. C. Kennedy, Ray Kesler, Olen Nance, J. Rosenbaum, Naomi Sherman, D. L. MacKay, R. E. Greenwood, and C. W. Trigg.*

No. 341. Proposed by *E. P. Starke*, Rutgers University.

The intersections of two parabolas whose axes are perpendicular are concyclic.

Solution by *W. T. Short*, Oklahoma Baptist University.

Let the axes be so selected that $y^2-4ax=0$ is the equation of one parabola. Then $(x-h)^2-4b(y-k)=0$ will be the equation of the other.

Thus
$$(x-h)^2-4b(y-k)+p(y^2-4ax)=0$$

is the pencil of conics determined by the four points of intersection of the two parabolas. If the parameter p is made unity, we have

$$(x-h-2a)^2+(y-2b)^2=4(ah+a^2+b^2-bk),$$

a circle whose center and radius are obvious.

Bernice Brevik noted that the converse is true. *M. S. Robertson* and *W. V. Parker* noted that the proposition may be generalized in the form: *The intersections of two conics are concylic if and only if an axis of symmetry of one conic is perpendicular to an axis of symmetry of the other.* A proof by the above method is not a difficult exercise.

Also solved by *Annie M. H. Christensen, K. W. Crain, D. L. MacKay*, and *C. W. Trigg.*

No. 343. Proposed by *Robert C. Yates*, Louisiana State University.

Given in the plane a point P and a line segment a with P not upon a. Outlawing the use of the compasses as a pair of dividers, construct with straightedge and compasses the circle with center at P and radius a.

Solution by *E. C. Kennedy*, Texas College of Arts and Industries.

Let AB be the given line segment a. Extend line AP to C, making $PC = AP$. This can be done under the rules of the problem. Then extend BP to D, making $PD = BP$. Extend CD to a point E such that $DE = CD$. Prolong BA to F, making $AF = BA$. Let FD and AE intersect in H. Then PH is the radius of the desired circle. The proof is obvious.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

If A and B are the end points of the given line segment, construct equilateral triangle PAC. Draw circle $A(AB)$, cutting CA prolonged in D and circle $C(CD)$ cutting CP prolonged in E. Circle $P(PE)$ is the required circle. The problem is Euclid, I, 2; see T. L. Heath, Thirteen Books of Euclid, Volume 1, p. 244.

Also solved by *Olen Nance*, *C. L. Steininger*, *C. W. Trigg*, and the *Proposer*.

PROPOSALS

No. 357. Proposed by *Cyril A. Nelson*, New Jersey College for Women.

Show how to arrange eight couples at bridge in such a fashion that in seven hands each man plays once with and once against each woman except his wife; each man plays once against each other man, and each woman plays once against each other woman.

Show that an analogous arrangement for six couples is not possible. Can a general statement be made concerning $2n$ couples?

No. 358. Proposed by *D. L. Mackay*, Evander Childs High School, New York.

Triangle ABC is divided into two parts: triangle DBE and quadrilateral $ADEC$, by the line DE. Construct a line which will bisect each of these parts.

No. 359. Proposed by "*A Wag*", Cincinnati, Ohio.

	1	3	6	10	15	21	28	36	45	55	66	78
	1	4	9	16	25	36	49	64	81	100	121	144
	1	8	27	64	125	216	343	512	729	1000	1331	1728
3)	3	15	42	90	165	273	420	612	855	1155	1518	1950
	1	5	14	30	55	91	140	204	285	385	506	650

What does this mean?

No. 360. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the sides of its two inscribed squares, construct the right triangle ABC.

No. 361. Proposed by *E. P. Starke*, Rutgers University.

Find all solutions of the equation, $2^x - 3^y = \pm 1$.

No. 362. Proposed by *N. A. Court*, University of Oklahoma.

Construct a tetrahedron so that its vertices shall lie on four concurrent lines, and the respectively opposite faces shall meet the corresponding lines in four non-coplanar preassigned points.

No. 363. Proposed by *Howard D. Grossman*, New York City.

The following columns of numbers (which may be indefinitely extended) are used in the old, well known trick of finding a number (as an age), given only the columns in which it appears.

Each positive integer may be expressed in just one way as a sum of powers of 2. Every integer in these columns is written once in each column headed by a power of 2 which appears in the sum for this integer. The integers in each column are written downward in order of increasing magnitude.

1	2	4	8
5	3	5	9
5	6	6	10
7	7	7	11
9	10	12	12
11	11	13	13
13	14	14	14
15	15	15	15

Now in each row there are four integers, some different, some alike. The integers in any row may be separated into sets, each set containing repetitions of the same integer. Let n_1 be the number of terms in the first set, n_2 the number in the second set, etc. Then all the sets $n_1, n_2, n_3, \cdots$ for all the rows constitute all the different par-

titions (including their permutations) of the number of columns, 4, into positive integers, viz., 1111, 211, 121, 31, 112, 22, 13, 4.

Prove that this is generally true regardless of the number of columns in the table.

No. 364. Proposed by *C. W. Trigg*, Los Angeles City College.

Through any point on the longer diagonal of a rhombus, which is formed from two equilateral triangles, lines are drawn parallel to the sides.

(1) The four intercepts and the extremities of the shorter diagonal are concyclic.
(2) The six points determine two congruent equilateral triangles with the same centroid.
(3) Find the radius of the circumcircle in terms of the segments of the diagonal.

No. 365. Proposed by *W. E. Byrne*, Virginia Military Institute, Lexington, Virginia.

By successive applications of the arithmetic process for square roots, the 2^nth root of M is to be obtained correct to k significant figures. How many significant figures should be retained in each of the successive operations?

So, the vast results obtained by Science are won by no mystical faculties, by no mental processes, other than those which are practiced by every one of us, in the humblest and meanest affairs of life. A detective policeman discovers a burglar from the marks made by his shoe, by a mental process identical with that by which Cuvier restored the extinct animals of Montmartre from fragments of their bones. Nor does that process of induction and deduction by which a lady, finding a stain of a peculiar kind upon her dress, concludes that somebody has upset the inkstand thereon, differ in any way, in kind, from that by which Adams and Leverrier discovered a new planet.

—From "*Science and Education*" by Thomas H. Huxley. Published by P. F. Collier and Son.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Teaching Arithmetic in the Elementary School. By Robert Lee Morton, Ohio University, Athens, Ohio. Volume I, Primary Grades. x+410 pages. Price $2.40. Volume II, Intermediate Grades. xii+538 pages. Price $2.75 Silver Burdett Company, 45 East 17th Street, New York, N. Y.

In these two volumes Dr. Morton has made available his interpretation of his own extensive experience and investigations in arithmetic; and "a serious effort has been made to bring together the findings, the conclusions, and the recommendations of other investigators and to organize them in a form which will be useful and meaningful to those interested." As a result, the elementary school teacher will find an abundance of helpful suggestions.

In Volume I, the author discusses in detail and evaluates for use in the classroom in the primary grades methods and procedures for developing an understanding of number; for teaching the fundamental combinations in addition, subtraction, multiplication, and division; and for obtaining understandings in Roman numerals, measures, fractions, and problem solving. In Volume II methods and procedures are discussed for developing in the intermediate grades the more advanced work in the fundamental processes, fractions, the elements of percentage, denominate numbers, and problem solving.

Prominent theories of instruction in arithmetic and the functions of arithmetic are analyzed. The "meaning" theory is stressed because it is believed to hold "much greater promise" than either the "drill" theory or the "incidental-learning" theory. "In brief, it is held that children should *understand* numbers before they learn to operate with them, that the *meaning* of the operations should be made clear before these operations are learned, that new steps and processes should be *discovered* (by the children) from their relationships to steps and processes already learned, that *drill does not make arithmetic meaningful* but is of value in fixing skills and maintaining them at a high level of usefulness, and that the *ideas and principles* of arithmetic rather than skill in computation are the means of developing in children the abililty to do quantitative thinking." "The arithmetic which gives due consideration to the informational, the sociological, and the phychological function, as well as to the computational function, is an indispensable part of the training of the young."

Theory and practice are discussed simultaneously and frequent references are made throughout to the results of investigations and experiments. Interesting discussions are centered around "the importance of an extensive and varied program of concrete number experience," when to begin arithmetic, the inadequacy of activities alone "as a means of providing a desirable education in number," individual differences, drill, the four fundamental processes as a means to an end where the end is problem solving, which method to use in subtraction, long division and short division, the relative importance of *accuracy* and *speed*, the place of tests, and the course of study. For the most part conclusions are conservative where there are controversial issues involved.

The material in both volumes has been organized and presented carefully and thoroughly. A set of questions and review exercises, a chapter test, and numerous selected

references follow each chapter. The books are well adapted both for general reading and for use as textbooks. Students, teachers, and administrators interested in the teaching of arithmetic in the first six grades will find these well written volumes a valuable contribution.

State Teachers College, DeKalb, Illinois EUGENE W. HELLMICH

Science in a Tavern. By Charles S. Slichter, University of Wisconsin Press, 1938. ix +186 pages. $3.00.

Science in a Tavern is a collection of essays written by Professor Charles S. Slichter, Dean Emeritus of the Graduate School of the University of Wisconsin. The various articles in the collection were originally presented as addresses; some before the Madison Literary Club, others to the Wisconsin Chapter of Phi Beta Kappa. One was read at a meeting of the Mathematical Association of America on the occasion of the two hundred and fiftieth anniversary of the publication of Newton's Principia. Most of them have appeared in print in various journals. It is indeed fortunate that the University of Wisconsin Press has assembled the essays and made them available in book form. They are charming, delightful, and instructive. They are both literary and scholarly. The author has wit, mellow philosophy, a wide experience with the higher education in America, and a profound appreciation of the role of science.

There are ten essays in all. The first of them is called The Royal Philosophers. It is an account of the organization by that name which started in England in the seventeenth century, founded the Royal Society of London in 1662, and which has continued through the centuries numbering among its members most of the great figures in English Science. The Royal Philosophers met in a London tavern, where amid food and conviviality, they discussed scientific research and natural philosophy. The debt of literature to the tavern is known by everyone who has read Shakespeare's plays or Keat's Lines on the Mermaid Tavern. The fact that science is also indebted is not so widely known. Professor Slichter develops the theme in the first two of his essays and refers to it again in the Phi Beta Kappa address entitled *The New Philosophy.* He shows that in the early days of British science the universities were not the patrons of research. Newton, while teaching at Cambridge, had to buy his laboratory materials out of his own pocket. Most of the scientists were amateurs who financed their own investigations, did their experiments at home, and later met in taverns to discuss them with their fellow enthusiasts. Thus "the Tavern was the Alma Mater of British Science".

Professor Slichter manufactures the word Polymath to denote a man who is at home in all the sciences. In the very interesting essay by the above name, he pays tribute to a famous list of Polymaths; among them Archimedes, Leonardo, Newton, Sir Christopher Wren, and in particular Lord Kelvin and Heaviside. He has amusing comments on the present age of specialization in this essay.

Principia and the Modern Age tells the story of Newton's Principia; how the Royal Society published it when Pepys of the famous diary was president; how Halley for whom a comet is named furnished the money; how it became a Magna Charta of man's power over his environment, and remade the ancient world into a modern world.

The essay on Industrialism is provocative and the reader will find it very interesting whether he agrees or not. This reader was delighted to come across an idea that she herself has expressed to some of her friends, namely that modern tendency to organize, which is an accompaniment of the industrial age, is merely a change of unit, a change from the individual to the institution.

The reviewer recommends that all teachers read the essay on Self Training of a Teacher. It has more value than a whole library of books on pedagogy.

Heaven's Highway is timely. The author depicts propaganda as the advertising material on the billboards of the highway of life. He tells youth of the dangers from following these billboards blindly. He warns against taking a short view. He tells young people not to judge events by the prejudiced motives of the present hour. He pays tribute to the altruism of youth which produces its so called radicalism, and asks for not less altruism, but the invention of a new form of altruism that will not evaporate at thirty. I'm sure young people like Dean Slichter.

Hunter College. JEWELL HUGHES BUSHEY.

Thomas Jefferson and Mathematics. (Second Edition). By David Eugene Smith. Scripta Mathematica. Volume I, Sept., 1932.

In this 16 page pamphlet, Dr. Smith tells of Jefferson's connection with mathematics from the time when, at William and Mary College, he studied mathematics under Dr. Samuel Small, of whom Jefferson wrote: "To his enlightened and affectionate guidance, I am indebted for everything". It was fifty years later, when he had retired from political life, that Jefferson made plans for resuming the study of mathematical astronomy. But, throughout the intervening half century, his correspondence frequently showed his continued interest in mathematics; and the main part of Dr. Smith's article is concerned with synopses and quotations from this correspondence. These range from a discussion of the use of geometrical models in the teaching of geometry and sound advice to David Rittenhouse as to the content of the *Transactions of the American Mathematical Society* to an estimate of the great French mathematicians of his time. One of his most remarkable accomplishments for one neither a specialist in, nor a recent student of, mathematics was his development (for the Lewis and Clarke expedition to the Northwest) of a method for calculating longitudes "without the aid of a time keeper". Though original with him, this method was afterward found to have been known in the previous century.

His announcement of his intention to resume the study of mathematical astronomy was shown by his correspondence to be no idle statement. In the same year, he wrote that "he had amused himself by calculating the hour lines of a horizontal dial for the latitude of the place",—a latitude which he seems himself to have determined. Soon after that he thanked Mr. Nathaniel Bowditch for a paper on the observation of a comet and another "on Dr. Stewart's formula for obtaining the sun's distance by the motion of the moon's apsides". Again, he commented on Napier's laws for the solution of right spherical triangles, noting that the French mathematicians had neglected these laws.

Later he was influential in founding the University of Virginia. He was one of its first trustees in 1814 and Rector in 1818 and 1819, and he was influential in giving to mathematics a high place in its course of study. His interest in the applications of mathematics and his devotion to the new institution are evidenced by the fact that "The work of laying out the lawn and the location of its buildings was done by himself with transit and level, and his note book describing this work is still extant".

For this and other reasons it is probable that the ideals he set led to the calling of Sylvester to the faculty, some twenty years after Jefferson's death. Jefferson was also influential in setting a high standard in mathematics at the newly founded West Point Military Academy and especially in causing the introduction of French rather than English texts.

When we turn from these thoughts of this early leader of the Republic to the present rather precarious position of our science, we may well wish that more of our politicians and statesmen or even our leaders in *education* had as sound an understanding of the principles of mathematics and as correct a judgment of its followers.

Eureka College. MARY W. NEWSON.

Introductory Mathematical Analysis. By Joel S. Georges and Jacob M. Kinney. The Macmillan Company, 1938, 605 pages.

This book is designed for a five hour two semester college course n mathematics and includes material from algebra, trignometry, analytic geometry, and the calculus. Since it furnishes the fundamental topics of these subjects it is adapted to students who will not pursue mathematics further, and also gives the foundation for more advanced mathematics. The simpler processes and the interesting applications of differential and integral calculus are given.

The functional concept idea is followed throughout the book. While there is not a universal agreement among teachers of mathematics that this the best plan to follow it seems to be very carefully worked out here and the book is to be commended to those teachers who want a boook written along this line. The course is developed systematically, clearly, and from a purely mathematical point of view. It contains a vast amount of well selected material. The pages are well filled but clearly written. The authors have furnished a cumulative review at the end of each chapter which furnishes a very valuable teaching aid.

Kansas State Agricultural College. W. T. STRATTON.

Plane Trigonometry. By Fred W. Sparks and Paul K. Rees. Prentice Hall, 1937. x+161+76 pages. $2.50.

This text covers all the essentials of plane trigonometry, including logarithms, graphs of the trigonometric functions, inverse trigonometric functions, and trigonometric equations. It also includes a chapter on complex numbers, and short appendices on versed functions, Mollweide's equations, and natural logarithms.

The explanations are made carefully and in detail. The derivation of the rule for expressing functions of any angle as functions of positive acute angles, and the derivation of the fundamental identities are especially good. However, no figure accompanies the definitions of angles of elevation and depression.

Frequent illustrative examples, with complete solutions, are given in the text. An adequate number of exercises and problems is included for each topic except that no numerical exercises are given on the use of the addition, double-, and half-angle formulas. Very few of the problems on logarithmic computation or solution of triangles require interpolation in their solution. The authors mention in their preface that they regard the elimination of interpolation as desirable, but the reviewer feels that more problems requiring interpolation should be provided for teachers who prefer not to omit practice in interpolation.

Another omission is noted in the chapter on the graphing of the trigonometric functions. The authors discuss in considerable detail the graphs of each of the six simple functions $\sin x, \cdots, \csc x$, but they do not mention how the graphs are modified for functions such as $y=2 \sin 3x$, although problems of this type are included. Also, there are several exercises on the graphing of the sum or difference of two trigonometric functions, but no explanation is given for the best procedure in constructing a graph of this type.

Answers to even-numbered problems are listed at the end of the book. Four-place tables of values of trigonometric functions at intervals of 10′, five-place logarithms of numbers, and five place logarithms of the trigonometric functions at intervals of 1′ are included.

Louisiana Polytechnic Institute. H. S. KALTENBORN.

Elementary Number Theory. By J. V. Uspensky and M. A. Heaslet. McGraw-Hill Book Company, 1939. x+484 pages.

This book is intended for undergraduate students. Some parts of it, in particular the chapter on Bernoullian numbers and the one on Liouville's methods, may, however, require rather slow and careful reading.

The authors have given an unusually clear, thorough, and complete treatment of the more strictly elementary parts of number theory. They state and prove a good many theorems not commonly included in beginning courses, in particular, a certain very useful theorem by Voronoi. There are many novel and original features in the proofs.

Brief biographical sketches of noted mathematicians are inserted in suitable places throughout the book. These will undoubtedly help to increase the interest of the reader in the subjects which these men investigated.

The appendices on magic squares, calendar problems, and card shuffling will appeal to beginners and to amateurs who enjoy mathematical novelties and puzzles.

The text contains nothing about continued fractions, analytic number theory, or geometric number theory, but these are seldom mentioned in beginning texts.

There are about 275 exercises and problems, some of which are very easy. There are, however, some which will be quite difficult for many undergraduates.

At the end of the book there are tables of indices and a table of primes less than 5000.

The type, quality of the paper, and the arrangement of the material are all excellent.

There are a few minor errors and misprints such as are bound to occur in any new text. The illustrative example at the foot of page 18 and the top of page 19 is incorrectly worked out. On page 35, near the foot of the page, $x=\alpha\beta t$ should read $x=a\beta t=(ab/d)t$. On page 71, near the foot of the page, $N=p_1p_2\cdots p_n$ should be replaced by $N=PP_1P_2\cdots P_n$. Problem 11 on page 83 should have the words "only if" omitted from its last sentence, since $N=8$ can be represented as a difference of two squares in only one way. In the center of page 258, we should replace

$$C_{2k}^{i}B_{k-i}\quad\frac{N^{2i+1}}{2i+1}\quad\text{by}\quad C_{2k}^{2i}B_{k-i}\quad\frac{N^{2i+1}}{2i+1}\ .$$

The reviewer feels that this book is an excellent text for a two-semester course in elementary number theory, and that, by suitable omissions, it could be used very nicely for a one-semester course. It should be included in the library of any department which offers such a course, even if some other book is adopted.

Tulane University. G. F. CRAMER.

Introduction to the Theory of Equations. By Louis Weisner. The Macmillan Company, New York, 1938. ix+188 pages. $2.25.

Chapter I begins with a discussion of complex numbers, defining them as real number pairs. A set of exercises that are well chosen, as they are throughout the book, illustrates the algebraic operations upon these numbers. A brief graphical treatment of them follows, and their representation in polar form leads to the ordinary method of finding the nth roots of complex numbers. In Chapter II the concept of a *field* is introduced, and this dominates all subsequent work. The set of all rational functions is treated as a field, and an algebraic equation is defined in terms of a field. The division

algorithm and the Euclidean algorithm are discussed, and then follow the topics *least common multiple, greatest common divisior, subfields, superfields, reducibility* and *irreducibility of a polynomial*, and the *unique factorization theorem.*

The title of Chapter III is *Further Properties of Polynomials in a Field.* In this the principal items are the relations between roots and coefficients, derivative of a polynomial (in a way, as he says, not involving continuity and limits), synthetic division, Taylor's series, and the construction of a polynomial satisfying certain conditions. About the middle of page 46 in Article 19 of this chapter occurs one of the few mistakes noted in the book. The expression

$$\frac{A(y)-A(x)}{y-x}$$

should be $a_0(y^{n-1}+y^{n-2}x+\cdots)$ instead of $a_0(y^{n-1}x+y^{n-2}x^2+\cdots)$.

In Chapter IV the field is restricted to $R(1)$, the field of rational numbers, and the usual method of determination of rational roots is given. In Chapter V the field is restricted to that of real numbers, which have the properties of being both ordered and compact. In discussing compactness, regular sequences of real numbers are introduced briefly, and then follow a discussion of continuity of a function of a real variable, Rolle's theorem, graphs of polynomials, isolation of the real roots by the methods of Sturm and Budan, and Descartes' rule of signs as a special case of Budan's theorem. Horner's and Newton's methods of approximating the real roots of an equation conclude the chapter.

Chapter VI is concerned with resultants, discriminants, and symmetric functions, attention being given to the degree and weight of the latter and to Newton's identities. The title of Chapter VII is *Algebraic Extensions of a Field.* The Cardan solution of the general cubic is given in terms of *radicals relative to a certain field*, and then follows the solution of the general quartic.

Chapter VIII is devoted to algebraically closed fields, a proof being given that the field of complex numbers is algebraically closed and thus proving the fundamental theorem of algebra. Chapter IX, the last chapter, treats at first construction by ruler and compasses. The field $R^{1/2}$ relative to R is introduced, and it is proved that a necessary and sufficient condition that a line segment of length x be constructibles by ruler and compasses from given line segments of lengths $a_1, a_2, \cdots, a_n$ is that x be a real element of the field $R^{1/2}(a_1, a_2, \cdots, a_n)$. The chapter closes with a discussion of the primitive nth roots of unity, irreducible polynomials, regular polygons inscribable in a given circle with ruler and compasses, and the method of inscribing a regular polygon of 17 sides. Following this chapter there are 115 miscellaneous exercises for review purposes and extra assignment.

On the whole, the book is well written and contains a good selection of exercises. The fact that almost all of the subject matter evolves from the idea of a field may afford a difficulty to some students, but for those well prepared this should be a good text.

University of North Carolina. ERNEST L. MACKIE.

Reviews and Examinations in Algebra. By Oswald Tower and Winfield Sides, Phillips Academy. D. C. Heath and Company, 1940. v+175 pages. $1.20.

Pages 1-101 contain 91 lists of approximately 1800 exercises and problems. These lists cover all topics offered in two full years of high school algebra, through *quadratics, progressions, the binomial theorem, variation, logarithms, and trigonometry.* The exercises

in each list are well graded as to difficulty, with very few repetitions. The final exercises of each list are excellent.

Pages 105-165 contain 12 recent *entrance* or *final examinations* given by West Point, Annapolis, The College Board, Phillips Academy, and other college preparatory schools, —also 11 well graded *tests* of 10 questions each, followed by a good list of 85 well selected supplementary review questions.

Pages 167-175 contain the *tables of squares and square roots, logarithms,* and *the trigonometric functions.*

We suggest that a copy of this book should be on the desk of every high school teacher who has a student preparing for the College Board Examination, an engineering school, or college work in the physical sciences and mathematics. It is a good modern source book of exercises and tests in algebra.

Evanston Township High School M. J. NEWELL

Fabre and Mathematics and Other Essays. By Lao Genevra Simons. Scripta Mathematica, 1939, New York. v+101 pages.

These essays are, according to the preface, "to some extent biographical and to some extent historical". Three of them have been published before. These are *Fabre and Mathematics, The Influence of French Mathematicians,* and *Short Stories in Colonial Geometry. The Interest of Alexander von Humbolt in Mathematics* appears for the first time in these pages. The essays on Fabre, von Humbolt and Colonial Geometry are enriched by very frequent quotations, as is essential considering the nature of the work.

Every student for whom mathematics has been spoiled by teachers who either did not know or did not like their subject should read the 24 pages on Fabre in this volume to see what mathematics may mean to a person with an adventurous spirit. The quotations from von Humbolt show him the possessor of extensive knowledge of the history of mathematics, in particular of the "numeration and notation of primitive tribes". In the "Kosmos", his survey of science, is found a justification of "the inclusion of mathematics in the development of the Cosmos". The third essay takes up several important French mathematicians and their influence at the end of the eighteenth century on mathematics teaching in American colleges. The last essay in the book considers the study of geometry in four colonial colleges, and seems to be the beginning of what should prove a decidedly interesting investigation, both mathematically and historically.

This volume might well be part of the library of any college mathematics club, and would make an excellent gift for young students interested in the subject of mathematics. The appearance of the book is very pleasing, and the errors are few.

Wellesley College MARION E. STARK

LITERATURE RECEIVED BY THE EDITORIAL BOARD DURING PERIOD APRIL 1, 1940 - JUNE 1, 1940 (VOLUME XIV)

(1) *Brief Course in Analytics.* By M. A. Hill and J. B. Linker. Henry Holt & Co., New York. ix+204 pages. 1940. $1.36.

(2) *Calculus.* By Charles K. Robbins and Neil Little. The Macmillan Co., New York. viii+398 pages. 1940. $3.25.

(3) *Elements of Analytic Geometry.* (Second Edition). By Clyde E. Love. The Macmillan Co., New York. xii+159 pages. 1940. $1.75.

(4) *Introduction to the Calculus.* By Arnold Dresden. Henry Holt & Co., New York. xii+428 pages. 1940. $3.40.

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To Our Colleagues In Mathematics

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A Non-Rotational Operator of Third Order Periodicity

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The Rectifying Curve

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The Drawing of Quadric Surfaces

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Index

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